Synthetic Lie Theory I

Synthetic Differential Geometry

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Summary

1. Smooth affine schemes

2. Open subsets, manifolds and sheaves

3. Weil bundles and the Kock-Lawvere axiom

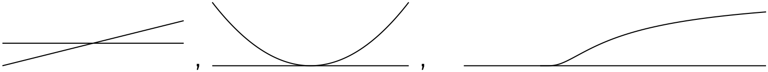
4. Lie brackets and formal group laws

Introduction

- In synthetic differential geometry we make rigorous the notions of infinitesimal object, infinitesimal action and infinitesimal transformation that are often useful heuristically in classical differential geometry. (c.f. methods of Newton and Leibniz.)
- At the end of the talk we will see how these three ideas are equivalent to each other.
- We will use algebra to rigorously justify the use of infinitesimals but in the end we will have a completely 'synthetic' theory.
- Finally we will begin to rephrase Lie theory in these terms.

Smooth Schemes

Consider the following intersections:



$$\{x = 0, y = 0\}\ ,\ \{y = x^2, y = 0\}\ ,\quad \{y = exp(-1/x), y = 0\}$$

Definition

The *category of C^∞-schemes* has as objects pairs $[n, I]$ where $n \in \mathbb{N}$ and I is a finitely generated ideal of $C^\infty(\mathbb{R}^n, \mathbb{R})$. The arrows

$$[n, I] \xrightarrow{f} [m, J]$$

are equivalence classes of smooth functions $f \in C^\infty(\mathbb{R}^n, \mathbb{R})$ such that:

- we identify $f \sim g$ iff $f \equiv g \pmod{I}$;
- for all $j \in J$ we have $jf \sim 0$.

Example

The arrows $[1, x^2] \to [1, -]$ are equivalence classes of functions $f : \mathbb{R} \to \mathbb{R}$ where $f \sim g$ iff $f \equiv g \pmod{x}^2$. Now using Hadamard's Lemma we see that

$$f(x) = f(0) + xf'(0) + x^2 g(x)$$

for some smooth function $g : \mathbb{R} \to \mathbb{R}$. Hence $f \sim f(0) + xf'(0)$ and so all arrows $[1, x^2] \to [1, -]$ are linear.

Example

The arrow $[1, x^3] \to [2, y^2 + x^2 - 1]$ defined by

$$x \mapsto (x, 1 - \frac{x^2}{2})$$

is well-typed because $(x, 1 - \frac{x^2}{2}) \equiv (sin(x), cos(x)) \pmod{x^3}$ and $sin^2(x) + cos^2(x) - 1 = 0$.

Open Subsets and Manifolds

Already in the category $\mathcal{C}$ of C^∞-schemes we have all smooth paracompact Hausdorff manifolds.

Definition

The *open subobject* U *of* $[n, I]$ *defined by* $\chi_U : \mathbb{R}^n \to \mathbb{R}$ is the subobject

$$[n+1, (I, \chi_U \cdot X_{n+1} - 1)] \xrightarrow{proj} [n, I]$$

which intuitively corresponds to the subset $\chi_U^{-1}(\mathbb{R} - \{0\}) \cap [n, I]$.

Definition

Let U be an open subset of $\mathbb{R}^n$ and $r : U \twoheadrightarrow M$ be a retract. Then

$$\iota M := [n+1, (\chi_U \cdot X_{n+1} = 1, r(x) = x)]$$

The function ι extends to a functor $\iota : Man \to \mathcal{C}$ that is full and faithful.

Completion of the Category of Affine C^∞-schemes

Problem: The category $\mathcal{C}$ of affine C^∞-schemes doesn't have all function spaces or quotients. We now apply a standard categorical technique to embed $\mathcal{C}$ in a category that does.

Definition

The *Dubuc coverage* $\mathcal{J}$ constists of the families of arrows

$$(U_i \rightarrowtail U)_{i \in I}$$

that are jointly surjective.

- First we embed $\mathcal{C}$ into its free cocompletion $\hat{\mathcal{C}}$;
- then we restrict to the subcategory $\mathcal{E}$ of $\hat{\mathcal{C}}$ consisting of the objects X such that

$$\begin{array}{ccc} \amalg_i U_i & \xrightarrow{\forall \phi} & X \\ \downarrow & \nearrow_{\exists ! \psi} & \\ U & & \end{array}$$

Jet Bundles and Weil Bundles

We write $D = [1, x^2]$. We have seen that arrows $D \to R$ are tangent vectors in R. Indeed we have an isomorphism

$$\iota M^D \cong \iota(TM)$$

We write $D_k = [1, x^{k+1}]$. Arrows $D_k \to R$ are of the form

$$d \mapsto a_0 + a_1 d + a_2 d^2 + ... + a_k d^k$$

and indeed $\iota M^{D_k} \cong \iota(T^k M)$ where $T^k M$ is the k-jet bundle.

Definition

A *Weil spectrum* D_W is an object of the form

$$[n, W] = \left[n, \bigwedge_{i=1}^{n} (x_1^{k_i} = 0) \wedge \bigwedge_{j=1}^{m} (p_j = 0) \right]$$

where $n, m \in \mathbb{N}$, $k_i \in \mathbb{N}_{>0}$ and p_j are polynomials in the x_i.

The Kock-Lawvere Axiom

Axiom (Kock-Lawvere)

For all Weil spectra D_W the R-algebra homomorphism

$$\frac{R[X_1, ..., X_n]}{(W)} \xrightarrow{\alpha} R^{D_W}$$

$$X_i \mapsto ((d_1, ..., d_n) \mapsto d_i)$$

is an isomorphism.

Example

If $n = 2$ and $W = (X_1^2, X_2^3, X_1X_2^2)$ then α is the R-algebra homomorphism that takes $(a_{00}, a_{10}, a_{01}, a_{11}, a_{02})$ to

$$(d_1, d_2) \mapsto a_{00} + a_{10}d_1 + a_{01}d_2 + a_{11}d_1d_2 + a_{02}d_2^2$$

Theorem

The Kock-Lawvere Axiom holds in the topos $\mathcal{E}$ by construction.

Lie Bracket of Vector Fields

Recall that a vector field is an arrow

$$M \xrightarrow{\hat{X}} M^D \iff D \times M \xrightarrow{X} M \iff D \xrightarrow{\check{X}} M^M$$

satisfying an extra condition (e.g. $\hat{X}(m)(0) = m$). Now given vector fields X, Y consider the arrow $\tau : D \times D \to M^M$ defined by:

$$\tau(d_1, d_2) = \check{Y}(-d_2) \circ \check{X}(-d_1) \circ \check{Y}(d_2) \circ \check{X}(d_1)$$

using the Kock-Lawvere axiom we see that for each $m \in M$

$$\begin{aligned} \tau(d_1, d_2)(m) &= m + \vec{a}_{10} d_1 + \vec{a}_{01} d_2 + \vec{a}_{11} d_1 d_2 \\ &= m + \vec{a}_{11} d_1 d_2 \end{aligned}$$

and so we can define the Lie bracket:

$$[X, Y](m)(d) = m + \vec{a}_{11} d$$

The Formal Group Law of a Lie Group

Definition

A formal group law F of dimension n is an n-tuple $(F_1, ..., F_n)$ of power series in the indeterminates $X_1, ..., X_n, Y_1, ..., Y_n$ such that

$$F(0, \vec{Y}) = \vec{Y} \ , \ F(\vec{X}, 0) = \vec{X} \text{ and } F(F(\vec{X}, \vec{Y}), \vec{Z}) = F(\vec{X}, F(\vec{Y}, \vec{Z}))$$

Example

Given a Lie group (G, μ, e) choose a trivialisation $U \ni e$ and $g, h \in U$ such that $\mu(g, h) \in U$. If $g = \vec{X}$ and $h = \vec{Y}$ in the local coordinates then $\mu(\vec{X}, \vec{Y})$ is a formal group law in $\vec{X}$ and $\vec{Y}$.

This depends on both a choice of trivialisation and that we have to restrict to those elements for which the multiplication is again in the trivialisation.

Transversal Pullbacks

Although we require that intersections in $\mathcal{E}$ are computed differently to intersections in *Man*, sometimes the two notions coincide.

Definition

Two smooth functions $f : M \to N$ and $g : L \to N$ with common codomain are *transversal* iff for all $m \in M$ and $l \in L$ such that $fm = gl$ we have that $(Df)_n$ and $(Dg)_l$ span the space $T_{fm}N$.

Theorem

The full and faithful embedding $\iota : Man \to \mathcal{E}$ preserves pullbacks of transversal pairs of arrows.