The Jet Part of a Category

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Summary

1. Formal Group Laws and Classical Lie Theory
2. The Jet Factorisation System
3. The Jet Part of a Category
4. The Coreflective Subcategory of Jet Categories

Classical Lie Theory

Recall that in classical Lie theory we have an adjunction

$$\mathit{FGpLaw} \underset{(-)_\infty}{\overset{(-)_{int}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathit{LieGrp}$$

When we restrict to the simply connected Lie groups:-

- Lie's second theorem says that $(-)_\infty$ is full and faithful;
- Lie's third theorem says that $(-)_\infty$ is essentially surjective.

We will generalise this situation in two main ways:-

- we will replace groups with categories;
- we will use synthetic differential geometry to view the local and global objects in the same category:

$$\mathit{Cat}_\infty(\mathcal{E}) \underset{(-)_\infty}{\underset{\longleftarrow}{\overset{\hookrightarrow}{\bot}}} \mathit{Cat}(\mathcal{E}) \underset{\hookleftarrow}{\overset{(-)_{int}}{\overset{\longrightarrow}{\bot}}} \mathit{Cat}_{int}(\mathcal{E})$$

Formal Group Laws

Definition 1.

A formal group law F of dimension n is an n-tuple $(F_1, ..., F_n)$ of power series in the indeterminates $X_1, ..., X_n, Y_1, ..., Y_n$ such that

$$F(0, \vec{Y}) = \vec{Y} \ , \ F(\vec{X}, 0) = \vec{X} \text{ and } F(F(\vec{X}, \vec{Y}), \vec{Z}) = F(\vec{X}, F(\vec{Y}, \vec{Z}))$$

Example 2.

Given a Lie group (G, μ, e) choose a trivialisation $U \ni e$ and $g, h \in U$ such that $\mu(g, h) \in U$. If $g = \vec{X}$ and $h = \vec{Y}$ in the local coordinates then $\mu(\vec{X}, \vec{Y})$ is a formal group law in $\vec{X}$ and $\vec{Y}$.

- We could have used the Campbell-Baker-Hausdorff formula but the above is more direct.
- We had to choose a trivialisation.
- This trivialisation was not closed under μ.
- These two problems disappear if we consider the infinitesimal neighbourhood of the identity instead of U.

The Jet Factorisation System

Definition 3.

The *spectrum of a Weil algebra D_W* is an object of the form

$$\{(x_1, \ldots, x_n) : \bigwedge_{i=1}^{n} (x_i^{k_i} = 0) \wedge \bigwedge_{j=1}^{m} (p_j = 0)\}$$

for $n, m \in \mathbb{N}_{\geq 0}$, $k_i \in \mathbb{N}_{>0}$ and p_j are polynomials in the x_i.

Definition 4.

An arrow $r : X \to Y$ in $\mathcal{E}/M$ is jet closed iff it is a monomorphism and for all Weil spectra D_W the square

$$\begin{array}{ccc} X^{M \times D_W \to M} & \xrightarrow{X^0} & X \\ \big\downarrow{\scriptstyle r^{M \times D_W \to M}} & & \big\downarrow{\scriptstyle r} \\ Y^{M \times D_W \to M} & \xrightarrow{Y^0} & Y \end{array}$$

is a pullback in $\mathcal{E}/M$.

The Jet Factorisation System

Definition 5.

An arrow $l : A \to B$ is jet dense in $\mathcal{E}/M$ iff for all jet closed r the square

$$\begin{array}{ccc} X^B & \xrightarrow{X^l} & X^A \\ \downarrow{\scriptstyle r^B} & & \downarrow{\scriptstyle r} \\ Y^B & \xrightarrow{Y^l} & Y^A \end{array}$$

is a pullback in $\mathcal{E}/M$.

Proposition 6.

The pair $(JetDense, JetClosed)$ defines a factorisation system. That is to say every arrow f in $\mathcal{E}$ factorises as $f = rl$ where l is jet dense and r is jet closed.

Composition in the Jet Part

Definition 7.
The jet part of a category has arrow space C_∞ given by the jet factorisation of the identity arrow

$$(M, 1_M) \xrightarrow{e_\infty} (C_\infty, s_\infty) \xrightarrow{\iota_C^\infty} (C, s)$$

Recall that for every arrow $f : C \to M$ in a topos $\mathcal{E}$ the pullback functor f^* has both a left and a right adjoint:

$$\mathcal{E}/C \;\underset{\Pi_f}{\overset{\Sigma_f}{\leftarrow f^* -}}\; \mathcal{E}/M$$

Moreover f^* preserves exponentials and for all X in $\mathcal{E}/C$ and A in $\mathcal{E}/M$ there is an isomorphism $\Pi_f(X^{f^*A}) \cong (\Pi_f X)^A$.

Lemma 8.

The functor Π_f preserves jet closed arrows:

$$\Pi_f\left(\begin{array}{ccc} X^{(C\times D_W,\pi_1)} & \xrightarrow{X^{(1_C,0)}} & X \\ \downarrow{\scriptstyle r^{(C\times D_W,\pi_1)}} & & \downarrow{\scriptstyle r} \\ Y^{(C\times D_W,\pi_1)} & \xrightarrow{Y^{(1_C,0)}} & Y \end{array}\right) \cong \begin{array}{ccc} \Pi_f(X)^{(M\times D_W,\pi_1)} & \xrightarrow{\Pi_f(X)^{(1_M,0)}} & \Pi_f(X) \\ \downarrow{\scriptstyle \Pi_f(r)^{(M\times D_W,\pi_1)}} & & \downarrow{\scriptstyle \Pi_f(r)} \\ \Pi_f(Y)^{(M\times D_W,\pi_1)} & \xrightarrow{\Pi_f(Y)^{(1_M,0)}} & \Pi_f(Y) \end{array}$$

Corollary 9.

The functor f^* preserves jet dense arrows.

Lemma 10.

The functor Σ_f preserves jet dense arrows.

Hence the following three arrows are jet dense in $\mathcal{E}/M$, $\mathcal{E}/G$ and $\mathcal{E}/M$ respectively.

$$(M, 1_M) \xrightarrow{e_\infty} (C, s_\infty)$$

$$t_\infty^*((M, 1_M) \xrightarrow{e_\infty} (C, s_\infty)) = (C, 1_C) \xrightarrow{t_\infty^*(e_\infty)} (C \;{}_{t_\infty}\!\times_{s_\infty} C, \pi_1)$$

$$\Sigma_{s_\infty}((C, 1_C) \xrightarrow{t_\infty^*(e_\infty)} (C\,{}_{t_\infty}\!\times_{s_\infty} C, \pi_1)) = (C, s_\infty) \to (C\,{}_{t_\infty}\!\times_{s_\infty} C, s_\infty \pi_1)$$

Proposition 11.

The subobject $C_\infty \rightarrowtail C$ induces a subcategory $\mathbb{C}_\infty \rightarrowtail \mathbb{C}$.

Proof.

The multiplication is given by the lift:

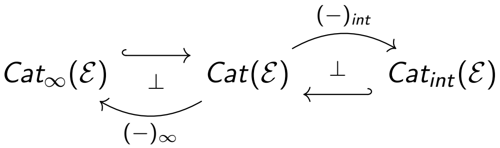
$$\begin{array}{ccc}
(C_\infty, s_\infty) & \xrightarrow{\quad 1_{C_\infty} \quad} & (C_\infty, s_\infty) \\
{\scriptstyle (1_{C_\infty}, e_\infty t_\infty)} \downarrow & \overset{\mu_\infty}{\nearrow} & \downarrow {\scriptstyle \iota_\infty} \\
(2_\times C_\infty, s_\infty \circ \pi_1) & \xrightarrow{\mu \circ (2_\times \iota_\infty)} & (C, s)
\end{array}$$

Definition 12.

A category $\mathbb{K}$ is called a *jet category* iff the arrow $\iota_{\mathbb{K}}^{\infty} : \mathbb{K}_{\infty} \to \mathbb{K}$ is an isomorphism.

Proposition 13.

The category $Cat_{\infty}(\mathcal{E})$ of jet categories is a mono-coreflective subcategory of $Cat(\mathcal{E})$.



The Jet Part of a Lie Group

Example 14.

Let $D_\infty = \bigcup_{k=1}^{\infty} D_k$ then $(R, +)_\infty = (D_\infty^n, +)$.

Example 15.

Since all Lie groups $\mathbb{G}$ are locally Euclidean we have that $\mathbb{G}_\infty = (D_\infty^n, \mu)$ for some $n \in \mathbb{N}$ and some multiplication μ.

- Now using the Kock-Lawvere axiom arrows

$$D_\infty^{2n} \xrightarrow{\mu} D_\infty^n$$

  are n-tuples of formal power series in indeterminates $X_1, ..., X_n, Y_1, ..., Y_n$ with values in nilpotent elements.
- But having values in nilpotent elements is equivalent to having zero constant term. Furthermore the unit and associativity laws for μ induce the structure of a formal group law on the n-tuple.