Infinitesimals in Lie Theory

Matthew Burke

University of Calgary, October 7, 2016

Summary

1. Introduction and Lie Groups
2. Lie Algebras and Lie's Theorems
3. Multi-object Lie Theory
4. Infinitesimals and Synthetic Lie Theory

Introduction to Lie Groups

Sophus Lie's original motivation to introduce Lie groups was to study the symmetries of solutions to differential equations.

Example 1.

If $g : \mathbb{R} \to \mathbb{R}$ is an integrable function then

$$\frac{dy}{dx} = g(x) \implies y = \int g(x)dx + c$$

Example 2.

$$\frac{dy}{dx} = y \implies y = ce^x$$

Smooth Manifolds

Slogan: A smooth manifold is a topological space that locally 'looks like' Euclidean space and globally 'fits together smoothly'.

Example 3.

- All open subsets U of $\mathbb{R}^n$.
- All graphs of smooth functions $f : \mathbb{R}^n \to \mathbb{R}^m$.
- The sphere S^2.
- Any smooth retract of an open subset (Whitney Embedding).

Definition 4.
A *Lie group* is a smooth manifold together with maps $\mu : G \times G \to G$ and $e : 1 \to G$ that satisfy the usual associativity and unit laws.

In this talk we consider Lie groups that are contained in $Mat(n, \mathbb{R})$.

Example 5.
The groups $GL(n, \mathbb{R})$, $SL(n, \mathbb{R})$, $O(n, \mathbb{R})$ and $SO(n, \mathbb{R})$.

Lie Algebras

‘Through the introduction and fundamental use of the infinitesimal transformations the theory of infinite continuous groups now takes on a surprising simplicity.’ **Sophus Lie**

- Lie used the infinitesimal transformations as ‘generators’ for his Lie groups;
- contrast between infinitesimal and infinite - in this case continuous;

‘as expedient the synthetic method is for discovery, as difficult it is to give a clear exposition on synthetic investigations’ **Sophus Lie**

- Grothendieck schemes and nilpotents;
- synthetic differential geometry of Lawvere and Kock which usually uses a Grothendieck topos;
- critically rejects the principle of the excluded middle;
- so we use constructive mathematics and intuitionistic logic;

The Nilradical and Jacobson Radical

Tangent vectors... In intuitionistic logic the following statements are not equivalent.

$$\begin{array}{ccccc} U(x) \vee \neg(x=0) & \longrightarrow & \neg(x=0) & \Longrightarrow & U(x) \\ \downarrow & & \downarrow & & \\ \neg U(x) \Longrightarrow (x=0) & \longrightarrow & \neg\neg(U(x) \vee (x=0)) & & \end{array}$$

hence we have four different kinds of field.

Definition 6.

The *Jacobson radical* $J(R) = \{x \in R : \forall u \in R.\ 1 - ux \textit{ is inv.}\}$.
The *nilradical N(R)* is the set of all the nilpotent elements of R.
$N(R) \subset J(R)$ because $(1-ud)(1+ud+(ud)^2+\ldots+(ud)^{k-1})$.
In a 'field of fractions':

$$\begin{aligned} 1 - ux \text{ is inv.} &\iff \neg(1 - ux = 0) \\ &\iff \neg(1 = ux) \\ &\iff \neg(x \text{ is inv.}) \\ &\iff \neg\neg(x = 0) \end{aligned}$$

Lie Algebras

Definition 7.

A *Lie algebra* is a real vector space V and a binary operation $[-,-] : A \times A \to A$ such that

- $[X, kY] = k[X, Y]$ and $[X, Y + Z] = [X, Y] + [X, Z]$;
- $[X, Y] = -[Y, X]$;
- $[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0$;

Example 8.

The vector space $Mat(n, \mathbb{R})$ with bracket $[X, Y] = XY - YX$.

Lie's Fundamental Theorems

If G is a Lie group then we can form a Lie algebra $T_e G$ that has as underlying vector space the tangent space at the identity.

$$LieAlg \underset{T_e}{\rightleftarrows} LieGp$$

In the case of a matrix Lie group the bracket is given by the commutator $[X, Y] = XY - YX$.

Example 9.

The Lie algebra of $GL(n, \mathbb{R})$ is $Mat(n, \mathbb{R})$. The Lie algebra of $SL(n, \mathbb{R})$ is the Lie algebra of traceless matrices.

Theorem 10.

- *(Lie II) if G and H are simply connected Lie groups and if $\phi : T_e G \to T_e H$ then there is a unique extension $\psi : G \to H$.*
- *(Lie III) if $\mathfrak{g}$ is a Lie algebra then there is a unique simply connected Lie group G such that $T_e G = \mathfrak{g}$.*

Lie Algebroids

Definition 11.

A *Lie algebroid* is a vector bundle $A \to M$ in *Man* together with a bundle homomorphism $\rho : A \to TM$ such that the space of sections $\Gamma(A)$ is a Lie algebra satisfying $(\forall X, Y \in \Gamma(A))(\forall f \in C^\infty(M))$:

$$[X, fY] = \rho(X)(f) \cdot Y + f[X, Y]$$

Example 12.

All Lie algebras and all tangent bundles.

Definition 13.

A *Lie groupoid* is a groupoid in *Man* such that the source and target maps are submersions.

In the multi-object setting, we still have a full and faithful functor

$$LieGpd_{sc} \xrightarrow{T_e} LieAlgd$$

but it is not essentially surjective.

- For every Lie algebroid there is a topological groupoid that is the 'obvious' candidate for the integral of the algebroid (its Weinstein groupoid) but there can be obstructions to putting a smooth structure on it - see [Crainic and Fernandes 2003].

Idea: Enlarge the category of smooth spaces:-

- Differentiable Stacks [Tseng and Zhu 2006].
- Using Synthetic Differential Geometry.

The Jet Part of a Lie Group

Example 14.

Let $D_\infty = \bigcup_{k=1}^{\infty} D_k$ then $(R, +)_\infty = (D_\infty^n, +)$.

Example 15.

Since all Lie groups $\mathbb{G}$ are locally Euclidean we have that $\mathbb{G}_\infty = (D_\infty^n, \mu)$ for some $n \in \mathbb{N}$ and some multiplication μ.

- Now using the Kock-Lawvere axiom arrows

$$D_\infty^{2n} \xrightarrow{\mu} D_\infty^n$$

  are n-tuples of formal power series in indeterminates $X_1, ..., X_n, Y_1, ..., Y_n$ with values in nilpotent elements.
- But having values in nilpotent elements is equivalent to having zero constant term. Furthermore the unit and associativity laws for μ induce the structure of a formal group law on the n-tuple.