Synthetic Lie Theory

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Contents

Abstract

Traditionally an infinitesimal neighbourhood of the identity element of a Lie group is studied indirectly by using an appropriately chosen algebraic structure such as a Lie algebra to represent it. In this thesis we use the theory of synthetic differential geometry to work directly with this infinitesimal neighbourhood and reformulate Lie theory in terms of infinitesimals. We show how to carry out this reformulation for the established generalisation of Lie theory involving Lie groupoids and Lie algebroids and make a further generalisation by replacing groupoids with categories. Our main result is a proof of Lie's second theorem in this context. Finally we show how our new constructions and definitions relate to the classical ones.

Statement of Originality

This thesis is submitted in fulfilment of the requirements of the degree of Doctor of Philosophy at Macquarie University. I certify the work in this thesis has not been submitted for a degree nor has it been submitted as part of the requirements for a degree to any other university or institution.

This thesis is my own work and any help that I have received in my research work and the preparation of this thesis has been appropriately acknowledged. To the best of my knowledge and ability the contents of this thesis are original except where specific reference is made to the work of others.

Matthew Burke

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Introduction

> "Through the introduction and fundamental use of the infinitesimal transformations, the theory of infinite continuous groups now takes on a surprising simplicity."
> Sophus Lie in [26]. Translation by D. H. Delphenich.

A Lie group is at once a smooth manifold and group. These two structures do not exist independently of each other: the operations of the underlying group preserve the smooth structure of the underlying manifold. This means that data contained in a small region around the identity element of the group may be transferred smoothly and systematically around the rest of the Lie group. In Lie theory we investigate how much of the information contained in a Lie group can be reconstructed using only data that is within an 'infinitesimal neighbourhood' of its identity element. Traditionally this infinitesimal neighbourhood is studied indirectly by using an appropriately chosen algebraic structure such as a Lie algebra to represent it.

Synthetic differential geometry is a theory which makes rigorous the notions of infinitesimal object, infinitesimal action and infinitesimal transformation that are often used heuristically in differential geometry. It is similar in spirit to the non-standard analysis of [32]. However all of the infinitesimals described in non-standard analysis are invertible but in synthetic differential geometry we have a rich collection of infinitesimal objects which always includes nilpotents and, depending on the model one chooses, germs of functions and the invertible infinitesimals also. The initial motivating idea behind this thesis was to use synthetic differential geometry to work directly with an infinitesimal neighbourhood of the identity element of a Lie group rather than its algebraic representation. It turns out that the nilpotent infinitesimals will play a central role in this reformulation.

In classical Lie theory we can in fact reconstruct all of the data in a Lie group $\mathbb{G}$ from data in its Lie algebra $\mathfrak{g}$ provided that $\mathbb{G}$ is simply connected. The precise description of this relationship between Lie groups and Lie algebras is encoded in the following fundamental theorems of Lie theory, which we refer to as Lie's first, second and third theorems respectively.

Theorem. *Let $\mathbb{G}_1$, $\mathbb{G}_2$ be Lie groups and $\mathfrak{g}_1$, $\mathfrak{g}_2$ be the corresponding Lie algebras. Then:*

1. *There is a bijection between connected Lie subgroups $\mathbb{G}_1 \subset \mathbb{G}_2$ and Lie subalgebras $\mathfrak{g}_1 \subset \mathfrak{g}_2$.*
2. *If $\mathbb{G}_1$ is simply connected then there is a bijection between Lie group homomorphisms $\mathbb{G}_1 \to \mathbb{G}_2$ and Lie algebra homomorphisms $\mathfrak{g}_1 \to \mathfrak{g}_2$.*
3. *Any Lie algebra is isomorphic to the Lie algebra of a Lie group.*

For the proofs of these theorems and the precise definitions of the terms involved we refer to Section 3.8 of [15]. It follows from these theorems that the category of simply connected Lie groups is equivalent to the category of Lie algebras. This thesis will mainly be concerned with Lie's second theorem but a few remarks will be made about Lie's third theorem in Section 4.2.

In fact this thesis will use synthetic differential geometry to reformulate the following established multi-object generalisation of Lie theory in terms of infinitesimals. The appropriate global object that generalises the concept of a Lie group is called a Lie groupoid which is defined as a groupoid internal to the category of smooth manifolds for which the source and target arrows are submersions. The condition on the source and target arrows is required to ensure that the object of composable arrows of a Lie groupoid is again a manifold. The algebraic structure that is used to represent an infinitesimal neighbourhood of the identity elements of a Lie groupoid is called a Lie algebroid. This is defined as a smooth vector bundle $A \to M$ together with a bundle homomorphism $\rho : A \to TM$ such that the space of sections $\Gamma(A)$ is a Lie algebra satisfying the Leibniz identity:

$$[X, fY] = \rho(X)(f) \cdot Y + f[X, Y]$$

Recall that in the classical setting Lie's second theorem says that the functor

$$LieGrp \xrightarrow{T_e} LieAlg$$

that takes a Lie group its Lie algebra is full and faithful when the domain is restricted to the subcategory of simply connected Lie groups. Similarly in the multi-object setting there is a functor T_e which assigns a Lie algebroid to every Lie groupoid. The result corresponding to Lie's second theorem still holds at this higher level of generality: when we restrict T_e to the subcategory of Lie groupoids whose source fibres are simply connected it becomes full and faithful.

The main goal of this thesis is to formulate and prove a version of Lie's second theorem using the infinitesimal objects provided by synthetic differential geometry. In fact we will be able to prove an even more general result stated in terms of categories rather than groupoids. Although the categories that will play the role of Lie groupoids will satisfy a condition which forces certain arrows to be invertible they nevertheless may still have some non-invertible arrows.

In fact there is another motivation for moving beyond classical differential geometry when working with Lie groupoids and Lie algebroids. Recall that in the classical setting Lie's third theorem assures us that any Lie algebra integrates to a simply connected Lie group. In categorical language this is equivalent to the statement that the functor T_e is essentially surjective. However the same is not true for Lie algebroids. Any Lie algebroid integrates to a topological groupoid, its Weinstein groupoid [7], but there can be obstructions to putting a smooth structure on it. This suggests that instead of working in the category of smooth manifolds it would be more convenient to work in a category in which one can simultaneously make sense of tangent vectors and Weinstein groupoids. In the paper [34] Tseng and Zhu show that the category of differentiable stacks is suitable for this purpose and in this thesis we will show that any well-adapted model of synthetic differential geometry (as defined in Chapter 1) is also.

We now proceed to describe in general terms the definitions and constructions that we will use in place of the those in classical Lie theory. We will delay the precise analysis of how our definitions generalise the classical ones until Chapter 5.

In order to prove Lie's second theorem we need to have a way of integrating infinitesimal data to macroscopic data. In the classical setting since all Lie groups are smooth manifolds we could appeal to well known results in differential geometry concerning the existence of solutions to smooth vector fields. In synthetic differential geometry however not all spaces are so well-behaved and so it will be necessary to impose a condition on our categories that mimics being able to find such solutions. It turns out that the integration that is required for Lie's second theorem is of a very specific kind and hence we require a much weaker condition than being able to solve all smooth vector fields. For example in [22] we find that the crucial transfer of information between the infinitesimal and the macroscopic is contained in the equivalence of a certain kind of path in the Lie algebra (which we will call A-paths) and paths in the Lie group starting at the identity element (which we will call G-paths). It turns out that when we have infinitesimals available to us the notion of A-path can be expressed straightforwardly as a certain internal functor. Since the definition of G-path doesn't involve a limit we can transfer it immediately to the synthetic setting. In order to isolate the categories for which these two types of path coincide we employ the theory of factorisation systems and its established relationship to completion operations as can be found in [16]. We will call such categories 'integral complete' categories.

Now we recall the way in which two specific algebraic structures are used to represent the part of a Lie group infinitesimally close to the identity element. First we describe the representation that uses linear (or first order) infinitesimals using a Lie algebra. Given a Lie group G we obtain a vector space by considering the tangent space T_eG at the identity element e. Then for each $v_e \in T_eG$ we define a vector field $v : G \to TG$ on the whole of G by

$$v_g = (DL_g)_e v_e$$

where L_g is left multiplication by g and D denotes the derivative. Now we can define a Lie bracket on T_eG by using the usual Lie bracket of vector fields. Second we describe a representation that uses nilpotent infinitesimals of all orders which is called a formal group law. Following [12] we define an n-dimensional formal group law F to be an n-tuple of power series in the variables $X_1, ..., X_n; Y_1, ..., Y_n$ with coefficients in $\mathbb{R}$ such that the equalities

$$F(\vec{X},\vec{0}) = \vec{X},\ \ F(\vec{0},\vec{Y}) = \vec{Y} \text{ and } F(F(\vec{X},\vec{Y}),\vec{Z}) = F(\vec{X},F(\vec{Y},\vec{Z})) \quad (1)$$

hold. We construct a formal group law from a Lie group G in two different ways. The first uses local coordinates. By using Proposition 1.117 in [18] we can assume without loss of generality that the multiplication of G is an analytic function. We start by choosing an open inclusion $\psi_e : C^n \rightarrowtail G$ where C^n is $(-1,1)^n$ in $\mathbb{R}^n$ such that $\psi_e(0)$ is the identity element e of G. Then we choose any $x, y \in im(\psi_e)$ such that xy is also in $im(\psi_e)$. Hence we have vectors $\vec{X} = \psi_e^{-1}(x)$ and $\vec{Y} = \psi_e^{-1}(y)$ in $\mathbb{R}^n$ and since G is analytic $\psi_e^{-1}(xy)$ can be expressed as an n-tuple of power series in the variables $(\vec{X},\vec{Y})$. It is easy to see that the equalities (1) hold. The second way to associate a formal group law to G is via its Lie algebra $\mathfrak{g}$. Given any Lie algebra one can form its Campbell-Baker-Hausdorff series (see Section IV.8 of Part I in [33]) which defines a formal group law. In fact the category of Lie algebras and formal group laws are shown to be equivalent in Theorem 3 of Section V.6 of Part 2 in [33]. When we form the infinitesimal part of a category $\mathbb{C}$ in Section 2.3.2 our construction will correspond to the part of a Lie group represented by its formal group law. In order to construct this subcategory $\iota_\infty : \mathbb{C}_\infty \rightarrowtail \mathbb{C}$ we will again use the theory of factorisation systems. We will describe a modification of the $(Epi, Mono)$-factorisation system which constructs, instead of just the image of an arrow, the smallest subobject containing the image and all appropriately defined 'infinitesimal perturbations' of this image. The categories for which ι_∞ is an isomorphism we will call 'jet categories'.

Using the jet and integral factorisation systems we can construct an adjunction

$$\begin{array}{ccc} & \xrightarrow{(-)_{int}} & \\ Cat_\infty(\mathcal{E}) & \perp & Cat_{int}(\mathcal{E}) \\ & \xleftarrow[(-)_\infty]{} & \end{array}$$

where $Cat_\infty(\mathcal{E})$ is category of jet categories and $Cat_{int}(\mathcal{E})$ is the category of integral complete categories. In this context Lie's second theorem says that when we restrict its domain to the full subcategory of $Cat_{int}(\mathcal{E})$ of objects satisfying certain connectedness conditions that will be defined in Chapter 3 the functor $(-)_\infty$ is full and faithful. Our proof may be split into two stages. The first uses the connectedness conditions to reduce the theorem to a statement about A-paths and G-paths. Then we use the relationship between A-paths and G-paths in integral complete categories to complete the proof.

The main original contributions of this thesis are:

- The construction of the jet part $\mathbb{C}_\infty$ of a category $\mathbb{C}$ in $\mathcal{E}$ in Section 2.3.2.
- The counterexample in Corollary 2.3.22 which shows that, despite being a category, the jet part $\mathbb{G}_\infty$ of a groupoid $\mathbb{G}$ in $\mathcal{E}$ is not necessarily a groupoid. Proposition 2.3.26 then identifies a condition on the arrow space of $\mathbb{G}$ that ensures that $\mathbb{G}_\infty$ is a groupoid.
- The construction of the Weinstein category $W\mathbb{C}$ of an arbitrary category $\mathbb{C}$ in $\mathcal{E}$ in Section 3.5.
- The definition of integral complete category in Section 2.3.3. The main result of the thesis is Theorem 4.1.6 which uses this integral completeness condition to prove the key lifting property involved in Lie's second theorem. Corollary 4.1.9 then proves Lie's second theorem in synthetic differential geometry.
- Example 4.2.1 which shows that Lie's third theorem does not hold for the formulation of Lie theory presented in this thesis.
- Proposition 5.1.11 shows that all source path connected Lie groupoids are $\mathcal{E}$-path connected; Proposition 5.2.10 shows that the jet part of a Lie groupoid is $\mathcal{E}$-path connected; Proposition 5.4.14 shows that every Lie groupoid is integral complete in the Cahiers topos; Corollary 5.3.12 shows that every simply connected Lie group is $\mathcal{E}$-simply connected.

Chapter 1

Synthetic Differential Geometry

> "First, note that the usual 'dynamical systems' involving for example the smooth actions of a monoid, if properly construed, will surely form a topos with all the virtues that that entails such as internal logic, good exactness, function space of 'dynamical systems', etc." F. William Lawvere in [24].

In this thesis we will use the theory of synthetic differential geometry to study the smooth spaces and functions that play a fundamental role in Lie theory. This means that rather than working in the category Man that has as its objects smooth finite dimensional paracompact Hausdorff manifolds (with or without boundary) and as its arrows smooth functions we will instead use a special type of topos called a well-adapted model of synthetic differential geometry. Every well-adapted model $\mathcal{E}$ admits a full and faithful embedding $\iota : Man \hookrightarrow \mathcal{E}$ and hence any results that only concern objects in the image of this embedding correspond to results about classical manifolds. There are several advantages of working in this enlarged category.

Firstly, in the category Man certain pullbacks do not exist. For example this means that we are unable to give the structure of a manifold to any algebraic set that has a singularity. In addition, although certain well behaved colimits constructed using atlases do exist in Man there are many useful colimits that do not exist. A simple example that we will use is the pushout

$I\ {}_1{+}_0\ I$ where $I = [0,1] \subset \mathbb{R}$ is the unit interval. In any topos all limits and colimits exist so we can we form all these objects in any well-adapted model. Secondly for manifolds M and N the function space M^N is rarely a finite dimensional manifold but since every topos is cartesian closed all function spaces exist in any well-adapted model.

Finally and most importantly, in any well-adapted model $\mathcal{E}$ of synthetic differential geometry we can make rigorous the notions of infinitesimal object, infinitesimal action and infinitesimal transformation. In particular there exists an object D in $\mathcal{E}$ which is the representing object for tangent vectors (or 1-jets) on a manifold. Intuitively we think of D as consisting of all points on the line that square to zero:

$$D = \{x \in R : x^2 = 0\}$$

where $R = \iota(\mathbb{R})$. In Man this would just be the terminal object $1 = \{0\}$ but in $\mathcal{E}$ it is not terminal: in fact the fundamental Kock-Lawvere axiom satisfied by every well-adapted model tells us that the map

$$R^2 \xrightarrow{\alpha} R^D$$

defined by

$$\alpha(a,b) = (d \mapsto a + bd)$$

is an isomorphism. More generally for any manifold M we have that $(\iota M)^D = \iota(TM)$ and the projection

$$(\iota M)^D \xrightarrow{\iota M^0} \iota M$$

can be given the structure of a vector bundle internal to $\mathcal{E}$. Furthermore synthetic differential geometry allows us to rigorously define higher order infinitesimal jets as well by using the representing objects

$$D_k = \{x \in R : x^{k+1} = 0\}$$

for $k \in \mathbb{N}_{>0}$ and arbitrary jets using the representing object

$$D_\infty = \bigcup_{k>0} D_k$$

An arrow $D_k \to M$ will be called a k-jet in M and an arrow $D_\infty \to M$ will simply be called a jet in M.

When we combine the existence of infinitesimal objects with the cartesian closed structure on $\mathcal{E}$ we can rigorously identify the notions of vector field, infinitesimal action and infinitesimal transformation that are often identified heuristically in classical differential geometry. To do this let us start with a section of the tangent bundle. That is to say an arrow

$$M \xrightarrow{\hat{\xi}} M^D$$

such that $M^0 \circ \xi = 1_M$. Using the hom-tensor adjunction once we obtain an arrow

$$D \times M \xrightarrow{\xi} M$$

such that $\xi(0, m) = m$ which is precisely the data of a pointed action of D on M. Using the hom-tensor adjunction again gives an arrow

$$D \xrightarrow{\check{\xi}} M^M$$

such that $\check{\xi}(0) = 1_M$ which we think of as a transformation infinitesimally close to the identity transformation. This perspective on vector fields and hence differential equations is treated in more detail in [20].

We will frequently use the internal logic of $\mathcal{E}$ when it is convenient to phrase the theory in terms of variables and propositions rather than commutative diagrams. In particular we will use the formulation of the internal logic for Grothendieck toposes that is described in Section VI.7 of [23].

The price we must pay for this convenient category is embodied in the following observation that is Lemma 1.1.1 in [14].

Lemma 1.0.1. *In a Boolean topos, no non-trivial ring can satisfy the Kock-Lawvere axiom.*

It is also the content of Exercise 1.1 in Part I of [19]. This means that the internal logic associated to any well-adapted model of synthetic differential geometry is intuitionistic and we must reject any arguments that rely on the principle of excluded middle. In particular, when working in the internal logic of $\mathcal{E}$ the strategy of proof by contradiction is invalid and the axiom of choice does not hold.

1.1 Models of Synthetic Differential Geometry

In order to motivate our construction of well-adapted models of synthetic differential geometry we recall some elementary concepts in classical algebraic geometry. Recall that in algebraic geometry we use an ideal in a polynomial ring

$$I \lhd \mathbb{R}[X_1, X_2, ..., X_n] = \mathbb{R}[\vec{X}]$$

to carve out a subset of $\mathbb{R}^n$ that consists of the points in the zero set of every $f \in I$:

$$Z(I) = \{\vec{x} \in \mathbb{R}^n : \forall f \in I.\ f(\vec{x}) = 0\}$$

In the other direction, given a subset $X \subset \mathbb{R}^n$ we can obtain an ideal $I \lhd \mathbb{R}[\vec{X}]$ that is the set of all polynomials that vanish on X:

$$\mathcal{O}(X) = \{f \in \mathbb{R}[\vec{X}] : \forall \vec{x} \in X.\ f(\vec{x}) = 0\}$$

The correspondence between subsets and ideals is imperfect in two ways. Firstly there are many subsets $X \subset \mathbb{R}^n$ that we cannot carve out using polynomial equations. As an example consider the Cantor set in $\mathbb{R}$. One simple way of removing this problem is to restrict our interest to a certain class of well-behaved sets: for example affine algebraic sets which are precisely the subsets that arise as zero sets of ideals of polynomial rings. Secondly there are many ideals that carve out the same subset. For example for each $k \in \mathbb{N}$ the ideals (x^k) carve out the subset $\{0\} \subset \mathbb{R}$. Thus there is an algebraic formula (in this case corresponding to nilpotency) that the geometry of algebraic sets does not recognise. One important feature of the theory of sheaves and schemes is that it redresses this imbalance. The construction of well-adapted models proceeds in an analogous way to the above theory.

In synthetic differential geometry we are interested in not only the subsets carved out by polynomials but also those carved out by arbitrary smooth functionals. Therefore we replace the polynomial rings $\mathbb{R}[\vec{X}]$ with algebraic structures called C^∞-rings that have enough structure to account for all smooth functions between Euclidean spaces. Now that we have enlarged the class of functions that we are interested in there are even more formulae that we cannot hope to distinguish between using the geometry of zero sets of smooth functions.

For example the ideals $(exp(-1/x^2))$ and (x) both carve out the subset $\{0\} \subset \mathbb{R}$. The extent to which we mitigate the imbalance between geometry and algebra corresponds to the model of synthetic differential geometry that we choose to use.

If we take our cue from the theory of schemes the natural way to construct a model would be to start with the opposite category of the category of C^∞-rings. It turns out that there is a natural Grothendieck coverage (corresponding to open inclusions of manifolds) that we can put on this category and hence we can form a sheaf category (or Grothendieck topos) $\mathcal{L}$. In $\mathcal{L}$ the imbalance between geometry and algebra has been overcome because the objects of the site are in bijection with ideals of C^∞-rings. However in synthetic differential geometry we are not only interested in generalising algebraic geometry but also finding a nice category for the study of differential geometry. Therefore it is important that the category that we work in has a full and faithful embedding of the category Man of smooth paracompact Hausdorff manifolds inside it. Unfortunately it turns out that $\mathcal{L}$ does not contain Man in a sufficiently nice manner and hence when choosing our site we must restrict to some subcategory of the opposite category of C^∞-rings. In Section 1.4 we will pick out five toposes that do contain a full and faithful embedding of Man. These different models are sensitive to different types of infinitesimal and local behaviour: the coarsest corresponds to taking zero sets as above and the most sensitive uses finitely generated C^∞-rings whose defining ideal is 'germ determined' as recalled in Definition 1.1.8.

First we must define the opposite category of the category of finitely generated C^∞-rings. This category will be similar to the syntactic category of the theory of C^∞-rings except we will allow arbitrary conjunction instead of finite conjunction when defining the objects. To construct a syntactic category we would usually begin from a signature and then build up the terms and formulae for a chosen fragment of logic. Then we would introduce a natural deduction system to specify axioms of the theory and finally use an established construction to form the syntactic category. However for the theory that we are interested in we can give a direct description based on Section II.1 of [28].

We will write $C^\infty(\mathbb{R}^n, \mathbb{R}^m)$ for the set of smooth functions $\mathbb{R}^n \to \mathbb{R}^m$. For a subset I of $C^\infty(\mathbb{R}^n, \mathbb{R})$ we will write

$$(I) = \{\Sigma_{j=1}^k r_j \phi_j : (r_j \in C^\infty(\mathbb{R}^n, \mathbb{R})) \wedge (\phi_j \in I) \wedge (k \in \mathbb{N})\}$$

for the ideal of $C^\infty(\mathbb{R}^n, \mathbb{R})$ generated by I.

Definition 1.1.1. The category of finitely generated affine C^∞-schemes $\mathcal{C}$ has as objects pairs $[n, I]$ where $n \in \mathbb{N}$ and I is a subset of $C^\infty(\mathbb{R}^n, \mathbb{R})$. An arrow

$$[n, I] \xrightarrow{\phi} [m, J]$$

of $\mathcal{C}$ is an equivalence class of smooth functions $\phi : \mathbb{R}^n \to \mathbb{R}^m$ such that for all $f \in (J)$ the function $f\phi$ is in (I); ϕ is equivalent to ϕ' iff componentwise $\phi_i - \phi'_i \in (I)$ for $i \in \{1, ..., m\}$.

Remark 1.1.2. The identification of arrows in the last condition of Definition 1.1.1 tells us how mapping out of $[n, I]$ is different from mapping out of $[n, -]$ where $-$ denotes the empty set of functions. The following example shows how many more arrows out of $D = [1, x \mapsto x^2]$ there are than out of the terminal object $1 = [0, -]$.

Lemma 1.1.3. *Every arrow $D = [1, x \mapsto x^2] \to R = [1, -]$ is equivalent to an arrow $R \to R$ of the form*

$$x \mapsto a + bx$$

where $a, b \in \mathbb{R}$.

Proof. Every $f : D \to R$ is an equivalence class of smooth functions $f : \mathbb{R} \to \mathbb{R}$. By a double application of Hadamard's Lemma we have:

$$f(x) = f(0) + xf'(0) + x^2 f_2(x)$$

for some smooth $f_2 : \mathbb{R} \to \mathbb{R}$. If we let g denote the linear function defined by $x \mapsto f(0) + xf'(0)$ then we see that $f - g = x^2 f_2$ and so since x^2 is one of the functions defining the domain we identify f and g. □

Remark 1.1.4. If we modified the definition of the category of finitely generated affine C^∞-schemes by only allowing finite subsets I when defining the objects we would instead obtain the syntactic category of C^∞-rings. This subcategory of $\mathcal{C}$ can be thought of as the (opposite) category of finitely presented

C^∞-rings and will be written $\mathcal{C}_{fp} \subset \mathcal{C}$. This is one of the four categories from which we will construct a well-adapted model of synthetic differential geometry.

For our coarsest site the underlying category will be the subcategory of $\mathcal{C}$ consisting of all objects $[n, I]$ which are determined by their 'points'. This corresponds to identifying all objects that carve out the same zero set. The site will be useful to us when we relate synthetic differential geometry to classical differential geometry but it will not itself give rise to a well-adapted model.

Definition 1.1.5. For a set of smooth functions $I \subset C^\infty(\mathbb{R}^n, \mathbb{R})$ we write

$$Z(I) = \{x \in \mathbb{R}^n : \forall f \in I.\ f(x) = 0\}$$

Then an object $[n, I]$ of $\mathcal{C}$ is point-determined iff

$$\forall g \in C^\infty(\mathbb{R}^n, \mathbb{R}).\ (\forall x \in Z(I).\ g(x) = 0) \implies g \in (I)$$

We will denote by $\mathcal{C}_{pt} \subset \mathcal{C}$ the full subcategory of $\mathcal{C}$ on the objects that are point-determined.

Notation 1.1.6. For every set of functions $I \subset C^\infty(\mathbb{R}^n, \mathbb{R})$ we can consider the subobjects of $[n, I]$ in $\mathcal{C}$. We will write $[n, I]_{pt}$ for the largest such subobject that is also an object of $\mathcal{C}_{pt}$. In more concrete terms $[n, I]_{pt} = [n, J]$ where J is the subset of smooth functions in $C^\infty(\mathbb{R}^n, \mathbb{R})$ that vanish at all points of $Z(I)$.

We will now pick out subcategories of $\mathcal{C}$ that will give rise to well-adapted models of synthetic differential geometry. The definitions are the direct translation of those in Theorem 4.2 of Part I in [28] (note that this theorem is listed as being in the fifth section of Part I on the contents page). The second subcategory $\mathcal{C}_{jet} \subset \mathcal{C}$ that we consider is sensitive enough that we have distinct objects for ideals differing only by nilpotent elements.

Definition 1.1.7. For a smooth function $f : \mathbb{R}^n \to \mathbb{R}$ we write $T_x(f)$ for the Taylor series of f at x. We write $P = \mathbb{R}[[X_1, ..., X_n]]$ for the ring of formal power series in n indeterminates. For a set of smooth functions $I \subset C^\infty(\mathbb{R}^n, \mathbb{R})$ we write

$$T_x(I) = \{\Sigma_{j=1}^k r_j T_x(\phi_j) : (r_j \in P) \wedge (\phi_j \in I)\}$$

for the ideal of P generated by I. Then an object $[n, I]$ is closed iff

$$\forall g \in C^\infty(\mathbb{R}^n, \mathbb{R}).\ (\forall x \in Z(I).\ T_x(g) \in T_x(I)) \implies g \in (I)$$

We will denote by $\mathcal{C}_{jet} \subset \mathcal{C}$ the full subcategory on the objects that are closed. We will write $[n, I]_{jet}$ for the largest subobject of $[n, I]$ in $\mathcal{C}$ that is also an object of $\mathcal{C}_{jet}$. In more concrete terms $[n, I]_{jet} = [n, J]$ where J is the subset of smooth functions g in $C^\infty(\mathbb{R}^n, \mathbb{R})$ such that for all $x \in Z(I)$ we have that $T_x(g) \in T_x(I)$.

Clearly we have that $\mathcal{C}_{pt} \subset \mathcal{C}_{jet}$. Geometrically a nilpotent of order k will correspond to 'infinitesimal jets' of order k. To aid this visualisation we remark that in $\mathcal{C}$ the intersection of the subobject $[2, y - x^{k+1}]$ of $[2, -]$ corresponding to a parabola with the subobject $[2, y]$ of $[2, -]$ representing the x-axis is the object $D_k = [1, x^{k+1}]$ representing infinitesimal k-jets. In the subcategory $\mathcal{C}_{pt}$ the object D_k does not exist and in fact $[1, x^{k+1}]_{pt} = [1, I] \cong [0, -]$ where I is the set of all smooth functions vanishing at 0. However in $\mathcal{C}_{jet}$ if k and l are distinct natural numbers then the objects $[1, x^{k+1}]_{jet}$ and $[1, x^{l+1}]_{jet}$ are not isomorphic.

The next subcategory $\mathcal{C}_{germ} \subset \mathcal{C}$ that we consider is sensitive enough that we have distinct objects for ideals that have the same points and jets but whose germs are different.

Definition 1.1.8. For a smooth function $f : \mathbb{R}^n \to \mathbb{R}$ we write $\mathbf{g}_x(f)$ for the equivalence class of functions that is the germ of f at $x \in \mathbb{R}^n$. Write G_x for the ring of germs of smooth functions at x. For a set of smooth functions $I \subset C^\infty(\mathbb{R}^n, \mathbb{R})$ we write

$$\mathbf{g}_x(I) = \{\Sigma_{j=1}^k r_j \mathbf{g}_x(\phi_j) : (r_j \in G_x) \wedge (\phi_j \in I)\}$$

for the set of finite linear combinations of germs of elements of I. Then an object $[n, I]$ is germ-determined iff

$$\forall g \in C^\infty(\mathbb{R}^n, \mathbb{R}).\ (\forall x \in Z(I).\ \mathbf{g}_x(g) \in \mathbf{g}_x(I)) \implies g \in (I)$$

We will denote by $\mathcal{C}_{germ} \subset \mathcal{C}$ the full subcategory on the objects that are germ-determined. We will write $[n, I]_{germ}$ for the largest subobject of $[n, I]$ in

$\mathcal{C}$ that is also an object of $\mathcal{C}_{germ}$. In more concrete terms $[n, I]_{germ} = [n, J]$ where J is the subset of smooth functions g in $C^\infty(\mathbb{R}^n, \mathbb{R})$ such that for all $x \in Z(I)$ we have that $\mathbf{g}_x(g) \in \mathbf{g}_x(I)$.

Clearly $\mathcal{C}_{jet} \subset \mathcal{C}_{germ}$. To visualise the difference between $\mathcal{C}_{jet}$ and $\mathcal{C}_{germ}$ consider the following examples. Let $f : \mathbb{R} \to \mathbb{R}$ be the smooth function defined as

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ exp(\frac{-1}{x}) & \text{if } x > 0 \end{cases}$$

and let $g : \mathbb{R} \to \mathbb{R}$ be

$$g(x) = exp(\frac{-1}{x^2})$$

Then the intersection of the subobjects $[2, y - g(x)]$ and $[2, y]$ of $[2, -]$ in $\mathcal{C}$ is $[1, g(x)]$. However $[1, g(x)]$ does not exist in $\mathcal{C}_{jet}$ and in fact $[1, g(x)]_{jet} = [1, J]$ where J is the set of all smooth functions which vanish along with all their derivatives at 0. Similarly the intersection of the subobjects $[2, \{y - f(x), y - f(-x)\}]$ and $[2, y]$ of $[2, -]$ in $\mathcal{C}$ is $[1, \{f(x), f(-x)\}]$. However in $\mathcal{C}_{jet}$ we have $[1, \{f(x), f(-x)\}]_{jet} = [1, J]$ again. By contrast in $\mathcal{C}_{germ}$ the objects $[1, g(x)]_{germ}$ and $[2, \{f(x), f(-x)\}]_{germ}$ are not isomorphic.

Finally we consider a subcategory of $\mathcal{C}_{jet}$ which recognises infinitesimal jets but only if they are consistently distributed along the whole space.

Definition 1.1.9. A Weil presentation of degree n is a finite set W of polynomials over $\mathbb{R}$ in indeterminates $X_1, ..., X_n$ such that for all $i \in \{1, ..., n\}$ there exists an integer $k_i \geq 1$ such that the polynomial $X_i^{k_i}$ is in W. Let R be a ring internal to some finitely complete category $\mathcal{C}$. Then the R-Weil spectrum D_W^R carved out by the Weil presentation W is the object described by the proposition

$$D_W^R = \{\vec{x} \in R^n : \ \forall f \in W.\ f(\vec{x}) = 0\}$$

in the internal logic of $\mathcal{C}$.

Remark 1.1.10. A $[1, -]$-Weil spectrum in $\mathcal{C}$ is an object of the form $[n, W]$ where W is a Weil presentation. For such spectra we drop the $[1, -]$ and write simply D_W and Weil spectrum. We will write $Spec(Weil)$ for the set of all objects of $\mathcal{C}$ that are Weil spectra.

Definition 1.1.11. The subcategory $\mathcal{C}_W \subset \mathcal{C}$ is the full subcategory on the objects of the form

$$[m, J] \times [n, W]$$

where J is the set of all functions vanishing on some embedded submanifold M of $\mathbb{R}^m$ and W is a Weil presentation.

Remark 1.1.12. The set of smooth functions $J = C^\infty(M, \mathbb{R})$ is point-determined. For a spectrum of a Weil algebra $[n, I]$ the set I is closed. Hence $\mathcal{C}_W \subset \mathcal{C}_{jet}$.

The next Theorem relates the subcategories $\mathcal{C}_{fp}$ and $\mathcal{C}_{germ}$. It is Theorem 6.3 in Part III of [19].

Theorem 1.1.13. *Every finitely presented object of $\mathcal{C}$ is germ-determined.*

We now put all of the categories in this section in a diagram that indicates the various inclusions between them:

$$\begin{array}{ccccc} \mathcal{C}_{pt} & \rightarrowtail & \mathcal{C}_W & \rightarrowtail & \mathcal{C}_{fp} \\ & & \downarrow & & \downarrow \\ & & \mathcal{C}_{jet} & \rightarrowtail & \mathcal{C}_{germ} \end{array}$$

It is a part of the larger diagram produced in Appendix 2 of [28] that includes several other subcategories which we will not use in this thesis.

1.2 The Full and Faithful Embedding

We first describe the full and faithful embedding

$$\iota : Man \hookrightarrow \mathcal{C}_{pt}$$

The following is Theorem 6.15 in [25].

Theorem 1.2.1. *Every smooth n-dimensional manifold with or without boundary admits a proper smooth embedding into $\mathbb{R}^{2n+1}$.*

Remark 1.2.2. Since every proper map is closed we see that if ι_M is a proper smooth embedding of an m-dimensional manifold M into $\mathbb{R}^{2m+1}$ given by Theorem 1.2.1 then $im(\iota_M)$ is closed.

The following is Lemma 2.26 in [25].

Lemma 1.2.3. *Suppose M is a smooth manifold with or without boundary, $A \subset M$ is a closed subset, and $f : A \to \mathbb{R}^k$ is a smooth function. For any open set U containing A, there exists a smooth function $\overline{f} : M \to \mathbb{R}^k$ such that $\overline{f}|_A = f$ and supp$(\overline{f}) \subset U$.*

With these strong results at our disposal we can define the embedding of Man into the category of point-determined objects.

Definition 1.2.4. For every manifold M in Man use Theorem 1.2.1 to choose a proper smooth embedding ι_M of M into $\mathbb{R}^{2m+1}$ where m is the dimension of M. Let $\phi : M \to N$ be a smooth function between manifolds M and N of dimensions m and n respectively. Then we define

$$Man \xrightarrow{\iota} \mathcal{C}_{pt}$$

to be the functor that takes ϕ to the arrow

$$[2m+1, I_{\iota_M}] \xrightarrow{\overline{\iota_N \phi}} [2n+1, I_{\iota_N}]$$

where I_{ι_M} and I_{ι_N} are the sets of all smooth functions that vanish on the image of ι_M. We have written $\overline{\iota_N \phi}$ for the lift of $\iota_N \phi$ along ι_M given by Lemma 1.2.3.

To see that ι respects composition we use the fact that I_{ι_M} and I_{ι_M} are point-determined. By the definition of arrows in $\mathcal{C}$ two smooth functions $\phi, \psi : \mathbb{R}^{2m+1} \to \mathbb{R}^{2n+1}$ that factor through ι_N define the same arrow from $[2m+1, I_M]$ to $[2n+1, I_N]$ in $\mathcal{C}$ iff for all $i \in \{1, ..., 2n+1\}$ we have $(\phi - \psi)_i \in I_M$. But by construction of I_M this is the same as the condition $\phi \iota_M = \psi \iota_M$. Now let

$$A \xrightarrow{f} B \xrightarrow{g} C$$

be smooth functions in Man. Then the equalities

$$\overline{\iota_C g}\,\overline{\iota_B f}\,\iota_A = \overline{\iota_C g}\,\iota_B f = \iota_C g f = \overline{\iota_C g f}\,\iota_A$$

imply that $\overline{\iota_C g}\,\overline{\iota_B f} = \overline{\iota_C g f}$ as required.

Lemma 1.2.5. *Let $\iota_M, \iota'_M : M \hookrightarrow \mathbb{R}^{2m+1}$ be two proper smooth embeddings of a manifold M of dimension m. Then $[2m+1, I_{\iota_M}] \cong [2m+1, I_{\iota'_m}]$ in $\mathcal{C}$.*

Proof. Let

$$\overline{\iota_M} : [2m+1, I_{\iota'_M}] \to [2m+1, I_{\iota_M}]$$

be the lift of ι_M along ι'_M and

$$\overline{\iota'_M} : [2m+1, I_{\iota_M}] \to [2m+1, I_{\iota'_M}]$$

the lift of ι'_M along ι_M. These two arrows are inverses in $\mathcal{C}$. Indeed

$$\overline{\iota_M}\,\overline{\iota'_M}\iota_M = \overline{\iota_M}\iota'_M = \iota_M$$

and

$$\overline{\iota'_M}\,\overline{\iota_M}\iota'_M = \overline{\iota'_M}\iota_M = \iota'_M$$

hence $\overline{\iota_M}\,\overline{\iota'_M} = 1_{[2m+1, I_{\iota_M}]}$ and $\overline{\iota'_M}\,\overline{\iota_M} = 1_{[2m+1, I_{\iota'_M}]}$ as required. □

Proposition 1.2.6. *The functor $\iota : Man \to \mathcal{C}_{pt}$ is full and faithful.*

Proof. To show that ι is full let M and N be smooth manifolds and

$$[2m+1, I_{\iota_M}] \xrightarrow{a} [2n+1, I_{\iota_N}]$$

be an arrow in $\mathcal{C}_{pt}$. Let $\alpha, \alpha' : \mathbb{R}^{2m+1} \to \mathbb{R}^{2n+1}$ be in the equivalence class a. Since a has domain $[2m+1, I_{\iota_M}]$ we have that $\alpha\iota_M = \alpha'\iota_M$. Since a has codomain $[2n+1, I_{\iota_N}]$ we have that $\alpha\iota_M$ factors through ι_N as $\iota_N\overline{a} = \alpha\iota_M$. This $\underline{a}$ is unique because ι_N is a monomorphism. Now $\iota(\underline{a}) = \overline{\iota_N\underline{a}}$ where $\overline{\iota_N\underline{a}}$ is the lift of $\iota_N\underline{a}$ along ι_M. Hence

$$\iota(\underline{a})\iota_M = \iota_N\underline{a} = \alpha\iota_M$$

and so $\iota(\underline{a}) = a$ as required.

To show that ι is faithful let $f, g : M \to N$ be smooth functions in Man such that $\iota(f) = \iota(g)$. Then lifting along ι_M tell us that $\overline{\iota_N f} = \overline{\iota_N g}$ whence

$$\iota_N f = \overline{\iota_N f}\iota_M = \overline{\iota_N g}\iota_M = \iota_N g$$

and since ι_N is a monomorphism we have that $f = g$ as required. □

Now that we have shown that the category Man embeds into the opposite category of point-determined C^∞-rings we immediately deduce that Man embeds into the categories $\mathcal{C}_W$, $\mathcal{C}_{jet}$, $\mathcal{C}_{fp}$, $\mathcal{C}_{germ}$ and $\mathcal{C}$ also. The next step will be to put the structure of a coverage $\mathcal{J}$ on all these subcategories of $\mathcal{C}$ for which the covering families are a mild generalisation of arbitrary open covers in Man. We will see that all of the resulting sites are subcanonical except for $(\mathcal{C}, \mathcal{J})$ itself and so we can embed them into the resulting sheaf toposes $\mathcal{E}_{pt}$, $\mathcal{E}_W$, $\mathcal{E}_{jet}$, $\mathcal{E}_{fp}$ and $\mathcal{E}_{germ}$. It turns out that $\mathcal{E}_{pt}$ isn't fine enough to properly model infinitesimal objects but the other sheaf toposes will be the well-adapted models for synthetic differential geometry that we use in this thesis.

Definition 1.2.7. The open subset $U_\chi \to [n, I]$ defined by the smooth function $\chi : \mathbb{R}^n \to \mathbb{R}$ is the subobject:

$$\big[n+1, I \cup \{\chi \cdot x_{n+1} - 1\}\big] \xrightarrow{proj} [n, I]$$

which intuitively speaking represents the intersection of the zero set determined by I with the open subset $\chi^{-1}(\mathbb{R} - \{0\})$.

Remark 1.2.8. If $[n, I]$ is either point-determined, jet-determined, finitely presented or germ-determined then U_χ is point-determined, jet-determined, finitely presented or germ-determined respectively.

Definition 1.2.9. The Dubuc coverage $\mathcal{J}$ on the category $\mathcal{C}$ consists of all families of open subsets $\{\iota_i : U_{\chi_i} \to [n, I]\}_i$ such that every global element $1 \to [n, I]$ factors through one of the ι_i.

Definition 1.2.10. Using the obvious restrictions we obtain sites

$$(\mathcal{C}_W, \mathcal{J}_W),\ (\mathcal{C}_{jet}, \mathcal{J}_{jet}),\ (\mathcal{C}_{fp}, \mathcal{J}_{fp}) \text{ and } (\mathcal{C}_{germ}, \mathcal{J}_{germ})$$

The well-adapted model of synthetic differential geometry $\mathcal{E}_{germ}$ is the sheaf topos

$$\mathcal{E}_{germ} \cong Sh(\mathcal{C}_{germ}, \mathcal{J}_{germ})$$

constructed from the site $(\mathcal{C}_{germ}, \mathcal{J}_{germ})$. Similarly for $\mathcal{E}_W$, $\mathcal{E}_{jet}$ and $\mathcal{E}_{fp}$. The topos $\mathcal{E}_W$ will be called the Cahiers topos and the topos $\mathcal{E}_{germ}$ will be called the Dubuc topos. In the sequel we will use the term 'smooth topos' to refer to an arbitrary well-adapted model.

Theorem 1.2.11. *The sites*

$$(\mathcal{C}_W, \mathcal{J}_W), (\mathcal{C}_{jet}, \mathcal{J}_{jet}), \mathcal{C}_{fp}, \mathcal{J}_{fp}) \text{ and } (\mathcal{C}_{germ}, \mathcal{J}_{germ})$$

are subcanonical.

Proof. For $(\mathcal{C}_W, \mathcal{J}_W)$ we refer to [10]. For $(\mathcal{C}_{jet}, \mathcal{J}_{jet})$ we refer to Lemma 2.2 in Chapter III of [28]. For $(\mathcal{C}_{fp}, \mathcal{J}_{fp})$ we refer to Corollary 3.2.22 in [35]. For $(\mathcal{C}_{germ}, \mathcal{J}_{germ})$ we refer to either Lemma 1.3 in Chapter III of [28] or Theorem 7.4 in Part III of [19]. □

Corollary 1.2.12. *There is a full and faithful embedding $\iota : Man \to \mathcal{E}$ where $\mathcal{E}$ is any of the toposes $\mathcal{E}_W$, $\mathcal{E}_{jet}$, $\mathcal{E}_{fp}$ and $\mathcal{E}_{germ}$.*

Proof. Follows immediately from Proposition 1.2.6 and Theorem 1.2.11. □

Remark 1.2.13. In Definition 1.2.9 we used arbitrary families of open sets. If instead we use the coverage $\mathcal{I}$ that consists of all finite families of open subsets $\{\iota_i : U_{\chi_i} \to [n, I]\}_i$ in $\mathcal{C}$ such that every global element $1 \to [n, I]$ factors through one of the ι_i then $(\mathcal{C}, \mathcal{I})$ is actually subcanonical and the associated sheaf category has a full and faithful embedding of Man inside it. However arbitrary open covers are considered important enough that the definition of well-adapted model rules out this site.

1.3 The Kock-Lawvere Axiom

Recall that by Lemma 1.1.3 arrows $[1, x^2] \to [1, -]$ in $\mathcal{C}$ are in bijection with linear functions $\mathbb{R} \to \mathbb{R}$ of the form $x \mapsto a + bx$ for some $a, b \in \mathbb{R}$. The Kock-Lawvere axiom generalises this property.

Definition 1.3.1. Let W be a Weil presentation. We define $A_W^{\mathbb{R}}$ to be the $\mathbb{R}$-algebra

$$\mathbb{R}[X_1, .., X_n]/(W)$$

where (W) denotes the ideal of $\mathbb{R}[X_1, ..., X_n]$ generated by W. Given a ring R in a topos $\mathcal{E}$, the Weil algebra A_W^R with presentation W is the internal R-algebra

$$R[X_1, ...X_n]/(W).$$

Example 1.3.2. If $n = 1$ and $W = \{X^2\}$ then $A_W^{\mathbb{R}}$ is the ring of dual numbers $\mathbb{R}[X]/(X^2)$ and A_W^R is its internal counterpart.

Example 1.3.3. If $n = 2$ and $W = \{X_1^2, X_2^3, X_1X_2^2\}$ then $A_W^{\mathbb{R}}$ consists of all polynomials of the form

$$a_{00} + a_{10}X_1 + a_{01}X_2 + a_{11}X_1X_2 + a_{02}X_2^2$$

with multiplication defined as usual subject to the relations imposed by W.

Let W be a Weil presentation and D_W^R be the R-Weil spectrum carved out by W as in Definition 1.1.9. Now in the pointwise R-algebra $R^{D_W^R}$ the elements p_i corresponding to

$$D_W^R \hookrightarrow R^n \xrightarrow{\pi_i} R$$

satisfy the relations in W for $i \in \{1, ..., n\}$ and so induce an R-algebra map

$$A_W^R = R[X_1, ..., X_n]/(W) \xrightarrow{\alpha} R^{D_W^R}$$

that sends X_i to p_i.

Example 1.3.4. If $n = 2$ and $W = \{X_1^2, X_2^3, X_1X_2^2\}$ then α is the R-algebra homomorphism defined by

$$(a_{00}, a_{10}, a_{01}, a_{11}, a_{02}) \mapsto \left((d_1, d_2) \mapsto a_{00} + a_{10}d_1 + a_{01}d_2 + a_{11}d_1d_2 + a_{02}d_2^2\right).$$

Axiom 1.3.5. The topos $\mathcal{E}$ satisfies the Kock-Lawvere axiom with respect to an internal ring R iff for every Weil presentation W the internal R-algebra homomorphism

$$A_W^R \xrightarrow{\alpha} R^{D_W^R}$$

defined above is an isomorphism.

Proposition 1.3.6. *The toposes $\mathcal{E}_W$, $\mathcal{E}_{jet}$, $\mathcal{E}_{fp}$ and $\mathcal{E}_{germ}$ satisfy the Kock-Lawvere axiom with respect to the internal ring $\iota(\mathbb{R})$.*

Proof. It will suffice to prove the result for the largest topos $\mathcal{E}_{germ}$ and for this proof we refer to Theorem 8.4 in Part III of [19]. □

An important property of spectra of Weil algebras is that they are 'atomic' objects of the topos. In short this says that they are small enough to only fit in one summand of any structure that we construct by glueing together other smaller structures. The next definition makes this idea precise.

Definition 1.3.7. An object X in a category $\mathcal{E}$ is atomic iff the endofunctor

$$\mathcal{E} \xrightarrow{(-)^X} \mathcal{E}$$

defined using the internal hom has a right adjoint.

Proposition 1.3.8. *The object D is atomic for all $D \in Spec(Weil)$ in any of the toposes $\mathcal{E}_W$, $\mathcal{E}_{jet}$, $\mathcal{E}_{fp}$ and $\mathcal{E}_{germ}$.*

Proof. This follows from the Example in Appendix 4 of [28]. □

1.4 Well-adapted Models

Recall that the pullback which defines the intersection of the two embedded submanifolds $\{(x, y) : y = x^2\}$ and $\{(x, 0)\}$ of $\mathbb{R}^2$ does exist in Man and is isomorphic to the terminal object. For our purposes we would like the intersection to be calculated in a similar way to the intersection number of these two algebraic sets: that is we would like to include not only the points that the two sets share but also the jets that they share. In this case both $\{(x, y) : y = x^2\}$ and $\{(x, y) : y = 0\}$ also have a 1-jet in common and hence we insist that the intersection in $\mathcal{C}$ is $[1, x^2]$. However sometimes when we calculate the intersection of two algebraic sets in Man it turns out to be the correct one. This is clearly the case when the two algebraic sets share no jets of any order. We generalise this idea slightly by recalling the following definition which is Definition 3.1 in Part III of [19].

Definition 1.4.1. A pair of maps $f_i : M_i \to N$ $(i = 1, 2)$ in Man with common codomain are said to be transversal to each other if for each pair of points $x_1 \in M_1$, $x_2 \in M_2$ with $f_1(x_1) = f_2(x_2)$ $(= y$ say), the images of $(df_i)_{x_i}$ $(i = 1, 2)$ jointly span T_yN as a vector space.

Now we are in a position to define what it means for a topos to be a well-adapted model of synthetic differential geometry.

Definition 1.4.2. A well-adapted model of synthetic differential geometry is a topos equipped with a full embedding $\iota : Man \to \mathcal{E}$ satisfying the following conditions. The functor ι preserves pullbacks of transversal pairs, preserves the terminal object and sends arbitrary open covers in Man to jointly epimorphic families in $\mathcal{E}$. We further insist that the topos $\mathcal{E}$ satisfies the Kock-Lawvere axiom with respect to the internal ring $\iota(\mathbb{R})$ and that all $\iota(\mathbb{R})$-Weil spectra are atomic.

Proposition 1.4.3. *The toposes $\mathcal{E}_W$, $\mathcal{E}_{jet}$, $\mathcal{E}_{fp}$ and $\mathcal{E}_{germ}$ are well-adapted models for synthetic differential geometry.*

Proof. Let $\mathcal{E}_\bullet$ denote any of $\mathcal{E}_W$, $\mathcal{E}_{jet}$, $\mathcal{E}_{fp}$ and $\mathcal{E}_{germ}$; let $\mathcal{C}_\bullet$ denote the corresponding site of definition. The existence of ι is Corollary 1.2.12. Since the Yoneda embedding $y : \mathcal{C}_\bullet \to \mathcal{E}_\bullet$ preserves all limits, to show that ι preserves transversal pullbacks it will suffice to show that the full embedding $Man \to \mathcal{C}_\bullet$ preserves transversal pullbacks. Since $\mathcal{C}_\bullet$ is a full subcategory of the category $\mathcal{C}$ of Definition 1.1.1 it will suffice to show that the embedding $Man \to \mathcal{C}$ preserves transversal pullbacks. This is Proposition 2.1 of Chapter II in [28]. It is clear that ι preserves the terminal object for all these toposes. Since open covers in Man map to covers in the Dubuc coverage in $\mathcal{C}^{op}$ they map to epimorphic families in $\mathcal{E}$. Finally $\mathcal{E}$ satisfies the Kock-Lawvere axiom with respect to $\iota(\mathbb{R})$ by Proposition1.3.6 and $\iota(\mathbb{R})$-Weil spectra are atomic by Proposition 1.3.8. □

In [14] the notion of well-adapted category is defined in Definition 1.3.20. Then in Theorem 1.3.27 it is shown that the sheaf topos generated by any well-adapted category with the Dubuc coverage is a well-adapted topos. Since all the sites

$$(\mathcal{C}_W, \mathcal{J}_W), (\mathcal{C}_{jet}, \mathcal{J}_{jet}), \mathcal{C}_{fp}, \mathcal{J}_{fp}) \text{ and } (\mathcal{C}_{germ}, \mathcal{J}_{germ})$$

are examples of well-adapted categories this means that Proposition 1.4.3 follows from this more general theory.

Chapter 2

Factorisation Systems

2.1 Generalities on Factorisation Systems

This section will recall some of the basic theory of orthogonal and weak factorisation systems in ordinary category theory as well as the theory of enriched orthogonal factorisation systems. We will first observe how to modify certain results about ordinary weak factorisation systems to obtain results about ordinary orthogonal factorisation systems and then how to modify results about ordinary orthogonal factorisation systems to obtain results about enriched orthogonal factorisation systems. Our goal will be to obtain two different methods that generate an enriched factorisation system in a topos $\mathcal{E}$ from a given class of arrows in $\mathcal{E}$. The first method will construct a factorisation system such that the right class is contained in the class of monomorphisms of $\mathcal{E}$ and the crucial construction involved will be a categorical limit. This method will be used to define the jet factorisation system in a well-adapted model in Section 2.2.1. The second method will use a transfinite construction called the small object argument which mostly utilises categorical colimits. This method will be used to define the integral factorisation system in the category of categories internal to a well-adapted model in Section 2.3.3.

2.1.1 Ordinary Factorisation Systems

First we will recall the elementary theory of ordinary orthogonal and weak factorisation systems. We will then see how to relate orthogonal and weak factorisation systems via the codiagonal construction. The final result of this comparison will be recorded as Corollary 2.1.21 which will be used in Section 2.1.2 to generate a factorisation system from a class of arrows that we want to be contained in the left class. The presentation of this established theory mainly uses [3] although the actual definitions of strong and weak orthogonality have been phrased in terms of hom-objects as in [8].

In this section $l : A \to B$ and $r : X \to Y$ will denote arrows in a category $\mathcal{E}$. The capital letters L and R will denote classes of arrows in $\mathcal{E}$. In the sequel the category $\mathcal{E}$ will either be a topos or a category of categories or groupoids in a topos. However we will initially develop the theory in more generality and impose extra conditions as we need them.

Definition 2.1.1. The arrow l is left orthogonal to r (written $l \perp r$) iff

$$\begin{array}{ccc} \mathcal{E}(B,X) & \xrightarrow{\mathcal{E}(l,X)} & \mathcal{E}(A,X) \\ \Big\downarrow{\scriptstyle \mathcal{E}(B,r)} & & \Big\downarrow{\scriptstyle \mathcal{E}(A,r)} \\ \mathcal{E}(B,Y) & \xrightarrow{\mathcal{E}(l,Y)} & \mathcal{E}(A,Y) \end{array}$$

is a pullback in Set.

Remark 2.1.2. To say that $l \perp r$ is equivalent to saying that for all commutative squares

$$\begin{array}{ccc} A & \xrightarrow{\phi} & X \\ \Big\downarrow{\scriptstyle l} & \nearrow{\scriptstyle \psi} & \Big\downarrow{\scriptstyle r} \\ B & \xrightarrow[\xi]{} & Y \end{array}$$

there exists an unique filler ψ.

Definition 2.1.3. Let S be a class of arrows in a category $\mathcal{E}$. Then the right orthogonal complement of S is the class

$$S^{\perp} := \{f \in \mathcal{E}^{\mathbf{2}} : \ \forall s \in S.\ s \perp f\}$$

and the left orthogonal complement is the class:

$$^{\perp}S := \{f \in \mathcal{E}^{\mathbf{2}} : \ \forall s \in S.\ f \perp s\}$$

Remark 2.1.4. It is immediate from the definition of orthogonality that $X^{\perp}$ is closed under limits in $\mathcal{E}^{\mathbf{2}}$ and $^{\perp}X$ is closed under colimits in $\mathcal{E}^{\mathbf{2}}$. In addition both classes are easily seen to be closed under composition, taking retracts and to contain the isomorphisms.

Definition 2.1.5. The pair (L, R) is a (orthogonal) prefactorisation system on $\mathcal{E}$ iff $L^{\perp} = R$ and $L = {}^{\perp}R$.

Definition 2.1.6. The pair (L, R) is a (orthogonal) factorisation system on $\mathcal{E}$ iff (L, R) is a prefactorisation system and (L, R)-factorisations exist: i.e. for every $f \in \mathcal{E}^{\mathbf{2}}$ there exist $l \in L, r \in R$ such that $f = r \circ l$.

Definition 2.1.7. A replete class S of arrows in $\mathcal{E}$ is one which satisfies the following condition: if $s \in S$, the arrows α and β are isomorphisms in $\mathcal{E}$ and the square

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle t} \\ C & \xrightarrow{\beta} & D \end{array}$$

commutes then the arrow $t \in S$ also.

Remark 2.1.8. The classes $S^{\perp}$ and ${}^{\perp}S$ are replete for any class S of arrows in $\mathcal{E}$.

The following Lemma provides sufficient conditions for a pair (L, R) to be a factorisation system that are often easier to check than the conditions in Definition 2.1.6.

Lemma 2.1.9. *The pair (L, R) is a factorisation system iff*

1. *the classes L and R are replete,*
2. *if $l \in L$ and $r \in R$ then $l \perp r$,*
3. *for every map f in $\mathcal{E}$, there exist $f_r \in R$ and $f_l \in L$ such that $f = f_r f_l$.*

Proof. It is immediate that all factorisation systems satisfy conditions 1–3 so it will suffice to show that conditions 1–3 imply that $L = {}^{\perp}R$ (then the equality $L^{\perp} = R$ follows by duality). Now condition 2 is equivalent to the inclusion $L \subset {}^{\perp}R$ and therefore it will suffice to show that $L \supset {}^{\perp}R$. So let $f \in {}^{\perp}R$ and consider the commutative diagram

$$\begin{array}{ccccc} A & \xrightarrow{1_A} & A & \xrightarrow{f_l} & I \\ \downarrow{\scriptstyle f_l} & & \downarrow{\scriptstyle f} & \nearrow{\scriptstyle m} & \downarrow{\scriptstyle f_r} \\ I & \xrightarrow{f_r} & B & \xrightarrow{1_B} & B \end{array}$$

where $f = f_r f_l$ is the (L, R)-factorisation given by condition 3. By hypothesis there is a unique filler m for the right hand square. But by condition 2 the filler 1_I for the whole rectangle is the unique such filler. Hence m is an isomorphism with inverse f_r and by condition 1 we have that $f \in L$ as required. □

Remark 2.1.10. Condition 2 is equivalent to either $L^{\perp} \supset R$ or $L \subset {}^{\perp}R$.

Definition 2.1.11. The object $Sq(l, r)$ of squares between l and r is defined by the pullback

$$\begin{array}{ccc} Sq(l,r) & \longrightarrow & \mathcal{E}(A,X) \\ \downarrow & & \downarrow{\scriptstyle \mathcal{E}(A,r)} \\ \mathcal{E}(B,Y) & \xrightarrow{\mathcal{E}(l,Y)} & \mathcal{E}(A,Y) \end{array}$$

in Set.

Remark 2.1.12. The set $Sq(l, r)$ is equivalently described as the set of all pairs of arrows (ϕ, ξ) such that the square

$$\begin{array}{ccc} A & \xrightarrow{\phi} & X \\ {\scriptstyle l}\downarrow & & \downarrow{\scriptstyle r} \\ B & \xrightarrow{\xi} & Y \end{array}$$

commutes.

Definition 2.1.13. The arrow l is weakly left orthogonal to r (written $l \pitchfork r$) iff the arrow $p : \mathcal{E}(B, X) \to Sq(l, r)$ induced by the pair $(\mathcal{E}(B, r), \mathcal{E}(l, X))$ is a split epimorphism.

Remark 2.1.14. If $l \pitchfork r$ then for all commutative squares:

$$\begin{array}{ccc} A & \xrightarrow{\phi} & X \\ {\scriptstyle l}\downarrow & \overset{\xi}{\nearrow} & \downarrow{\scriptstyle r} \\ B & \xrightarrow{\psi} & Y \end{array}$$

there exists a (not necessarily unique) filler ξ. The converse holds if we assume the axiom of choice because under this assumption every epimorphism in Set is split.

Definition 2.1.15. Let S be a class of arrows in a category $\mathcal{E}$. Then the right weakly orthogonal complement of S is the class

$$S^{\pitchfork} := \{f \in \mathcal{E}^{\mathbf{2}} : \ \forall s \in S.\ s \pitchfork f\}$$

and the left weakly orthogonal complement is the class:

$$^{\pitchfork}S := \{f \in \mathcal{E}^{\mathbf{2}} : \ \forall s \in S.\ f \pitchfork s\}$$

Next, we prove some results that relate strong and weak orthogonality. Suppose from now on that $\mathcal{E}$ has all pushouts.

Definition 2.1.16. The codiagonal $\delta^o(l)$ of l is the arrow $B\,{}_l{+}_l\,B \to B$ induced by the pair $(1_B, 1_B)$:

$$\begin{array}{ccccc} A & \xrightarrow{l} & B & & \\ \downarrow{\scriptstyle l} & & \downarrow{\scriptstyle \iota_2} & \searrow{\scriptstyle 1_B} & \\ B & \xrightarrow{\iota_1} & B\,{}_l{+}_l\,B & & \\ & \searrow{\scriptstyle 1_B} & & \overset{\delta^o(l)}{\searrow} & \\ & & & & B \end{array}$$

If Σ is a class of arrows in $\mathcal{E}$ then we write $\delta^o(\Sigma) = \{\delta^o(l) : l \in \Sigma\}$.

The following Lemma follows immediately from the universal property of $B\,{}_l{+}_l\,B$.

Lemma 2.1.17. *An arrow $(\xi_1, \xi_2) : B\,{}_l{+}_l\,B \to X$ has a lift*

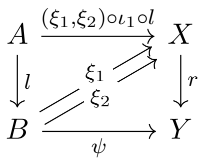
$$\begin{array}{ccc} B\,{}_l{+}_l\,B & \xrightarrow{(\xi_1,\xi_2)} & X \\ {\scriptstyle \delta^o(l)}\downarrow & \overset{\xi}{\nearrow} & \\ B & & \end{array}$$

along $\delta^o(l)$ iff $\xi_1 = \xi_2$, in which case $\xi = \xi_1 = \xi_2$.

Lemma 2.1.18. *The square*

$$\begin{array}{ccc} B\,{}_l{+}_l\,B & \xrightarrow{(\xi_1,\xi_2)} & X \\ {\scriptstyle \delta^o(l)}\downarrow & & \downarrow{\scriptstyle r} \\ B & \xrightarrow{\psi} & Y \end{array} \tag{2.1}$$

commutes iff the following diagram is serially commutative:



(2.2)

Proof. The universal property of $B\ {}_l{+}_l\ B$ tells us that the commutativity of (2.1) is equivalent to the equation $(\psi, \psi) = (r \circ \xi_1, r \circ \xi_2)$ which is in turn equivalent to the bottom triangles of (2.2) commuting. The top triangles of (2.2) commute tautologically. $\square$

Proposition 2.1.19. $l \bot r \iff (l \pitchfork r) \wedge (\delta^o(l) \pitchfork r)$

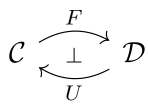
Proof. $\implies$: Suppose (2.1) commutes. Then by Lemma 2.1.18 we have that (2.2) is serially commutative. But by the hypothesis that $l \perp r$ we obtain that $\xi_1 = \xi = \xi_2$ and so by Lemma 2.1.17 we see that (2.1) is filled by ξ. Note that $r\xi = \psi$ because (2.2) is serially commutative. $\impliedby$: Suppose that (2.2) is serially commutative. Then by Lemma 2.1.18 we have that (2.1) commutes. Now by the hypothesis $\delta^o(l) \pitchfork r$ (2.1) has a filler so by Lemma 2.1.17 we have that $\xi_1 = \xi = \xi_2$. $\square$

Corollary 2.1.20. *If* $l \bot r$ *then* $\delta^o(l) \bot r$.

Proof. By Lemma 2.1.19 we have that $l \perp r \implies \delta^o \pitchfork r$. Now it remains to show that fillers of (2.1) must be unique. But this is clear because $\delta^o(l)$ is a (split) epimorphism. $\square$

Corollary 2.1.21. *If* Σ *is a class of arrows in* $\mathcal{E}$ *then* $\Sigma \cup \delta^o(\Sigma) \subset {}^{\perp}(\Sigma^{\perp})$.

Lemma 2.1.22. *Let* $(L_{\mathcal{C}}, R_{\mathcal{C}})$ *and* $(L_{\mathcal{D}}, R_{\mathcal{D}})$ *be factorisation systems on categories* $\mathcal{C}$ *and* $\mathcal{D}$ *respectively. Let*



be an adjunction. Then for all arrows l *in* $\mathcal{C}$ *and* ρ *in* $\mathcal{D}$ *we have that*

$$Fl \perp \rho \iff l \perp U\rho$$

Proof. First we see that $l \perp U\rho$ iff the square

$$\begin{array}{ccc} UX^B & \xrightarrow{UX^l} & UX^A \\ \big\downarrow{\scriptstyle U\rho^B} & & \big\downarrow{\scriptstyle U\rho^A} \\ UY^B & \xrightarrow{UY^l} & UY^A \end{array}$$

is a pullback in Set. Since $F \dashv U$ this is equivalent to

$$\begin{array}{ccc} X^{FB} & \xrightarrow{X^{Fl}} & X^{FA} \\ \big\downarrow{\scriptstyle \rho^{FB}} & & \big\downarrow{\scriptstyle \rho^{FA}} \\ Y^{FB} & \xrightarrow{Y^{Fl}} & Y^{FA} \end{array}$$

being a pullback in Set. But this is precisely the statement that $Fl \perp \rho$. □

2.1.2 Generating Ordinary Factorisation Systems

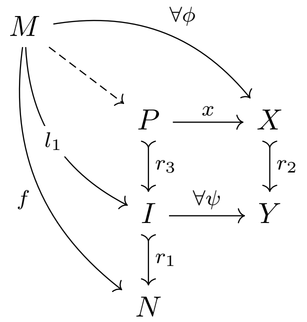
Now we describe two methods for generating ordinary factorisation systems from a class of arrows. The fundamental constructions will be carried out in the unenriched setting and then in the next section we will prove a 'bootstrapping' Lemma that enables us to obtain two enriched factorisation systems as corollaries of the constructions in this section.

The first method generates a factorisation system for which the right class is contained in the class of monomorphisms of $\mathcal{E}$. This will allow us to construct the factorisation using a limit (in particular an intersection). The following Proposition gives sufficient conditions on $\mathcal{E}$ and a set of arrows R in $\mathcal{E}$ for us to generate a factorisation system on $\mathcal{E}$ that has right class R. It is Lemma 3.1 of [6] where a sketch of the proof is given. Our treatment follows Proposition 7.1 in [27] where a full proof is given.

Proposition 2.1.23. *Let R be a class of arrows in a category $\mathcal{E}$. Suppose that R is contained in the class of monomorphisms, is closed under composition and contains all the isomorphisms. Suppose that the pullback of an arrow in R along an arbitrary arrow in $\mathcal{E}$ exists in $\mathcal{E}$ and is again in R. Suppose further that all intersections of arrows in R exist in $\mathcal{E}$ and are again in R. Then $({}^{\perp}R, R)$ is a factorisation system on $\mathcal{E}$.*

Proof. We will check the conditions of Lemma 2.1.9. To show that $({}^{\perp}R, R)$-factorisations exist let $f : M \to N$ be an arrow in $\mathcal{E}$ that we want to factorise.

Consider the diagram $D : \mathcal{I} \to \mathcal{E}/N$ where $\mathcal{I}$ is the full subcategory of $\mathcal{E}/N$ on all arrows in R with codomain N that f factors through. Let $r_1 : I \rightarrowtail N$ be the limit of this diagram (i.e. the intersection) and $l_1 : M \to I$ be the induced factorisation of f through the limit. We will show that for all $r_2 \in R$ we have $l_1 \perp r$ and hence $l_1 \in {}^{\perp}R$. So let ϕ and ψ be arrows in $\mathcal{E}$ such that the square $MIYX$ in the diagram



commutes and let r_3 be the pullback of r_2 along ψ. We need to show that the square $MIYX$ has a diagonal lift. Since R is closed under pullback and composition we have that $r_1 \circ r_3$ is an arrow in R that f factors through. Therefore r_3 is an isomorphism because I was defined as the intersection of all such arrows. So now $x \circ r_3^{-1}$ is the required lift which is the unique lift because r_2 is a monomorphism. It is immediate that the other conditions of Lemma 2.1.9 are satisfied. □

The second method uses a standard transfinite construction called the small object argument to generate a factorisation system. This method is carried out in the following Proposition 2.1.28 which gives sufficient conditions on $\mathcal{E}$ and a set of arrows L in $\mathcal{E}$ that allow us to generate a factorisation system on $\mathcal{E}$ that has left class containing (but not necessarily equal to) L. It is Theorem 4.1 in [3] and in our proof we also refer to the treatment of the small object argument in Section 4.5 of that paper.

Notation 2.1.24. Let α be a non-zero ordinal. Then we write $[\alpha]$ for the poset of all ordinals smaller than α and 0 for the initial object of $[\alpha]$. Let Σ be a class of arrows in a category $\mathcal{E}$. Then a transfinite string of composable arrows in Σ is a functor

$$[\alpha] \xrightarrow{D} \mathcal{E}$$

that preserves colimits such that for all $\beta + 1 < \alpha$ the natural map

$$D(\beta) \to D(\beta + 1)$$

is in Σ. We say that Σ is closed under transfinite composition iff for every D that is a transfinite string of composable arrows in Σ such that the colimit $colim_{\beta<\alpha} D(\beta)$ exists in $\mathcal{E}$ the induced map

$$D(0) \xrightarrow{\nu} colim_{\beta<\alpha} D(\beta)$$

is also in Σ.

Definition 2.1.25. A class of arrows Σ in a category $\mathcal{E}$ is closed under cobase change iff for all $\sigma \in \Sigma$ and all pushout squares

$$\begin{array}{ccc} A & \xrightarrow{f} & C \\ \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle u_1} \\ B & \xrightarrow{u_2} & D \end{array}$$

the arrow $u_1 \in \Sigma$ also.

Definition 2.1.26. Let Σ be a class of arrows in a category $\mathcal{E}$. Then $Sat(\Sigma)$ is the closure of this class under:

1. adding isomorphisms,
2. transfinite composition,
3. cobase change,
4. retracts.

We say that a class of arrows Σ is saturated iff $Sat(\Sigma) = \Sigma$.

Lemma 2.1.27. *The class of arrows $^{\perp}R$ is saturated for any R.*

Proof. It is clear that $^{\perp}R$ contains all the isomorphisms in $\mathcal{E}$ and that it is stable under cobase change. To see that it is closed under transfinite composition let $r \in R$, let the functor D be a transfinite string of composable arrows in $^{\perp}R$ and let ϕ and ξ be arrows in $\mathcal{E}$ making the outer square of

$$\begin{array}{ccc} D(0) & \xrightarrow{\phi} & C \\ \downarrow{\scriptstyle \nu} & \overset{\psi_\alpha}{\nearrow} & \downarrow{\scriptstyle r} \\ colim_{\beta<\alpha} D(\beta) & \xrightarrow{\xi} & E \end{array}$$

commute. For $\beta < \alpha$ we define the arrows $\psi_\beta : D(\beta) \to C$ by transfinite induction. For the base case we set $\psi_0 = \phi$. For successor ordinals we define $\psi_{\beta+1}$ as the unique filler of the square

$$\begin{array}{ccc} D(\beta) & \xrightarrow{\psi_\beta} & C \\ \downarrow & \overset{\psi_{\beta+1}}{\nearrow} & \downarrow r \\ D(\beta+1) & \xrightarrow{\xi\iota} & E \end{array}$$

where ι is the natural inclusion into $colim_{\gamma<\alpha} D(\gamma)$ and for limit ordinals β we define ψ_β as the unique map

$$colim_{\gamma<\beta} D(\gamma) = D(\beta) \xrightarrow{\psi_\beta} C$$

so that

$$D(\delta) \xrightarrow{\iota_\delta} colim_{\gamma<\beta} D(\gamma) \xrightarrow{\psi_\beta} C = \psi_\delta$$

for all $\delta < \beta$. Hence the ψ_α that we require is $colim_{\beta<\alpha} D(\beta)$.

To see that ${}^\perp R$ is closed under retract suppose that $g \in {}^\perp R$ and the square

$$\begin{array}{ccccc} A & \overset{m_1}{\rightarrowtail} & C & \overset{r_1}{\twoheadrightarrow} & A \\ \downarrow f & & \downarrow g & & \downarrow f \\ B & \underset{m_2}{\rightarrowtail} & D & \underset{r_2}{\twoheadrightarrow} & B \end{array}$$

commutes where $r_1 m_1 = 1_A$ and $r_2 m_2 = 1_B$. Further let ϕ and ξ make the square

$$\begin{array}{ccc} A & \xrightarrow{\phi} & X \\ \downarrow f & & \downarrow r \\ B & \xrightarrow{\xi} & Y \end{array}$$

commute. Then there exists an unique $\psi_1 : D \to X$ such that $\psi_1 g = \phi r_1$ and $r\psi_1 = \xi r_2$. But then the lift that we require is $\psi_1 m_2$. Indeed $r\psi_1 m_2 = \xi r_2 m_2 = \xi$ and $\psi_1 m_2 f = \psi_1 g m_1 = \phi r_1 m_1 = \phi$ as required. □

Proposition 2.1.28. *Let Σ be a set of arrows in a locally presentable category $\mathcal{E}$. Then*

$$(L, R) = (Sat(\Sigma \cup \delta^o(\Sigma)), \Sigma^\perp)$$

is a factorisation system on $\mathcal{E}$.

Proof. We check that the conditions of Lemma 2.1.9 hold. It is immediate that both $Sat(\Sigma \cup \delta^o(\Sigma))$ and $\Sigma^\perp$ are replete. To show that (L, R)-factorisations exist we refer to Section 4.5 of [3]. Thus it remains to check that

$$Sat(\Sigma \cup \delta^o(\Sigma))^\perp \supset \Sigma^\perp$$

Now Corollary 2.1.21 tells us that

$$\Sigma \cup \delta^o(\Sigma) \subset {}^\perp(\Sigma^\perp)$$

and so

$$Sat(\Sigma \cup \delta^o(\Sigma)) \subset {}^\perp(\Sigma^\perp)$$

because ${}^\perp(\Sigma^\perp)$ is saturated. Therefore

$$Sat(\Sigma \cup \delta^o(\Sigma))^\perp \supset {}^\perp(\Sigma^\perp)^\perp = \Sigma^\perp$$

as required. □

2.1.3 Enriched Factorisation Systems

We begin by recalling the theory of enriched (orthogonal) factorisation systems. Then we show how to relate enriched and ordinary factorisation systems using the operation of closure under (co)tensor. Once this relation is established we can immediately generalise Propositions 2.1.23 and 2.1.28 to obtain two methods of generating an enriched factorisation system from a class of arrows. They will be used to construct the jet factorisation system in Section 2.2.1 and the integral factorisation system in Section 2.3.3 respectively.

Since we followed [8] by defining the orthogonality of arrows in an ordinary category in terms of hom-objects it is now straightforward to give the general definition of enriched orthogonality and hence enriched factorisation system. We will work analogously to the treatment of weak enriched factorisation systems in [31] but will mainly make use of the account of (orthogonal) enriched factorisation systems that is [27]. We refer to [17] for the basic concepts of enriched category theory.

In this section $l : A \to B$ and $r : X \to Y$ will be arrows in the underlying category $\mathcal{E}_0$ of a $\mathcal{V}$-category $\mathcal{E}$. The capital letters L and R will denote classes of arrows in $\mathcal{E}$. In the sequel we will take $\mathcal{V}$ to be a smooth topos and $\mathcal{E}$ will be equal to one of the categories $Gpd(\mathcal{V})$, $Cat(\mathcal{V})$ or $\mathcal{V}$ itself. However for the moment we will only impose extra conditions on $\mathcal{E}$ and $\mathcal{V}$ as we require them.

Definition 2.1.29. The arrow l is left $\mathcal{V}$-orthogonal to r (written $l \perp_{\mathcal{V}} r$) iff

$$\begin{array}{ccc} \mathcal{E}(B,X) & \xrightarrow{\mathcal{E}(l,X)} & \mathcal{E}(A,X) \\ \downarrow{\scriptstyle \mathcal{E}(B,r)} & & \downarrow{\scriptstyle \mathcal{E}(A,r)} \\ \mathcal{E}(B,Y) & \xrightarrow{\mathcal{E}(l,Y)} & \mathcal{E}(A,Y) \end{array}$$

is a pullback in $\mathcal{V}$.

Definition 2.1.30. Let S be a class of arrows in $\mathcal{E}_0$. Then the right $\mathcal{V}$-orthogonal complement of S is the class:

$$S^{\perp_{\mathcal{V}}} := \{f \in \mathcal{E}_0^{\mathbf{2}} : \ \forall s \in S.\ s \perp_{\mathcal{V}} f\}$$

and the left $\mathcal{V}$-orthogonal complement of S is the class:

$${}^{\perp_{\mathcal{V}}}S := \{f \in \mathcal{E}_0^{\mathbf{2}} : \ \forall s \in S.\ f \perp_{\mathcal{V}} s\}$$

Definition 2.1.31. The pair (L, R) is a $\mathcal{V}$-prefactorisation system on $\mathcal{E}$ iff $L^{\perp_{\mathcal{V}}} = R$ and $L = {}^{\perp_{\mathcal{V}}}R$.

Definition 2.1.32. The pair (L, R) is a $\mathcal{V}$-factorisation system on $\mathcal{E}$ iff (L, R) is a prefactorisation system and (L, R)-factorisations exist: i.e. for every $f \in \mathcal{E}_0^{\mathbf{2}}$ there exist $l \in L, r \in R$ such that $f = r \circ l$.

Definition 2.1.33. Let v be an object of $\mathcal{V}$ and X an object of $\mathcal{E}$. Then the cotensor X^v is an object of $\mathcal{E}$ for which there is an isomorphism

$$\mathcal{E}(A, X^v) \cong \mathcal{V}(v, \mathcal{E}(A, X))$$

in $\mathcal{V}$ that is natural in A. For $f : X \to Y$ in $\mathcal{E}$ then the cotensor

$$X^v \xrightarrow{f^v} Y^v$$

of the arrow f by v is the arrow induced by $\mathcal{V}(v, \mathcal{E}(A, f))$ using the Yoneda Lemma. Dually the tensor $v \otimes X$ of X with v is an object of $\mathcal{E}$ for which there is an isomorphism

$$\mathcal{E}(v \otimes X, B) \cong \mathcal{V}(v, \mathcal{E}(X, B))$$

in $\mathcal{V}$ that is natural in B. For $f : X \to Y$ in $\mathcal{E}$ then the tensor

$$v \otimes X \xrightarrow{v \otimes f} v \otimes Y$$

of the arrow f by v is the arrow induced by $\mathcal{V}(v, \mathcal{E}(f, B))$ using the Yoneda Lemma.

The following Lemma is Proposition 5.4 in [27].

Lemma 2.1.34. *Let $l : A \to B$ and $r : X \to Y$ be arrows in $\mathcal{E}$ such that the cotensor r^v exists for all $v \in \mathcal{V}$. Then we have that*

$$(\forall v \in \mathcal{V}.\ l \perp r^v) \iff l \perp_{\mathcal{V}} r$$

Proof. The statement $(\forall v \in \mathcal{V}.\ l \perp r^v)$ is equivalent to the statement that

$$\begin{array}{ccc} \mathcal{E}(B, X^v) & \xrightarrow{\mathcal{E}(l, X^v)} & \mathcal{E}(A, X^v) \\ \downarrow{\scriptstyle \mathcal{E}(B, r^v)} & & \downarrow{\scriptstyle \mathcal{E}(A, r^v)} \\ \mathcal{E}(B, Y^v) & \xrightarrow{\mathcal{E}(l, Y^v)} & \mathcal{E}(A, Y^v) \end{array}$$

is a pullback in *Set* for all $v \in \mathcal{V}$. By definition of cotensor this is equivalent to the following square being a pullback in *Set*:

$$\begin{array}{ccc} \mathcal{V}(v, \mathcal{E}(B, X)) & \xrightarrow{\mathcal{V}(v, \mathcal{E}(l, X))} & \mathcal{V}(v, \mathcal{E}(A, X)) \\ \downarrow{\scriptstyle \mathcal{V}(v, \mathcal{E}(B, r))} & & \downarrow{\scriptstyle \mathcal{V}(v, \mathcal{E}(A, r))} \\ \mathcal{V}(v, \mathcal{E}(B, Y)) & \xrightarrow{\mathcal{V}(v\mathcal{E}(l, Y))} & \mathcal{V}(v, \mathcal{E}(A, Y)) \end{array}$$

for all $v \in \mathcal{V}$. By the Yoneda Lemma this is equivalent to the square:

$$\begin{array}{ccc} \mathcal{E}(B, X) & \xrightarrow{\mathcal{E}(l, X)} & \mathcal{E}(A, X) \\ \downarrow{\scriptstyle \mathcal{E}(B, r)} & & \downarrow{\scriptstyle \mathcal{E}(A, r)} \\ \mathcal{E}(B, Y) & \xrightarrow{\mathcal{E}(l, Y)} & \mathcal{E}(A, Y) \end{array}$$

being a pullback in $\mathcal{V}$ which is the definition of $l \perp_{\mathcal{V}} r$. □

Corollary 2.1.35. *Suppose that for all $v \in \mathcal{V}$ and all arrows $r \in R$ the cotensor r^v exists. Let $\bar{R}$ be the closure of R under cotensor with arbitrary objects of $\mathcal{V}$ (i.e. $\bar{R}$ is the smallest class containing R such that for all $x \in \bar{R}$ and $v \in \mathcal{V}$ the arrow x^v is also in $\bar{R}$) . Then ${}^{\perp_{\mathcal{V}}}R = {}^{\perp}\bar{R}$.*

Definition 2.1.36. An arrow m in $\mathcal{E}_0$ is a $\mathcal{V}$-monomorphism iff for all $A \in \mathcal{E}$ the arrow $\mathcal{E}(A, m) : \mathcal{E}(A, M) \to \mathcal{E}(A, N)$ is a monomorphism in $\mathcal{V}$. An arrow f in $\mathcal{E}_0$ is a strong $\mathcal{V}$-epimorphism iff $f \perp_{\mathcal{V}} m$ for all $\mathcal{V}$-monomorphisms m in $\mathcal{E}$.

Remark 2.1.37. Let $\mathcal{M}$ be the class of $\mathcal{V}$-monomorphisms in a $\mathcal{V}$-category $\mathcal{E}$ that has all coproducts. Then $\mathcal{M} = \mathcal{Q}^{\perp_{\mathcal{V}}}$ where $\mathcal{Q}$ is the set of all fold maps $\nabla : A + A \to A$ for $A \in \mathcal{E}$.

Corollary 2.1.38. *Let Λ be a class of arrows in the underlying category $\mathcal{E}_0$ of a $\mathcal{V}$-category $\mathcal{E}$ which has all tensors, cotensors and coproducts. Let $\mathcal{M}$ be the $\mathcal{V}$-monomorphisms in $\mathcal{E}$ and suppose that all intersections of arrows in $\mathcal{M}$ exist in $\mathcal{E}$. Then*

$$(L, R) = ({}^{\perp_{\mathcal{V}}}(\Lambda^{\perp_{\mathcal{V}}} \cap \mathcal{M}), \Lambda^{\perp_{\mathcal{V}}} \cap \mathcal{M})$$

is a $\mathcal{V}$-factorisation system on $\mathcal{E}$.

Proof. We first show that (L, R) is a $\mathcal{V}$-prefactorisation system. Since

$$\Lambda^{\perp_{\mathcal{V}}} \cap \mathcal{M} = \Lambda^{\perp_{\mathcal{V}}} \cap \mathcal{Q}^{\perp_{\mathcal{V}}} = (\Lambda \cup \mathcal{Q})^{\perp_{\mathcal{V}}}$$

it follows that

$$({}^{\perp_{\mathcal{V}}}R)^{\perp_{\mathcal{V}}} = R$$

as required. The existence of (L, R)-factorisations follows immediately from the fact that R is closed under cotensors (and so ${}^{\perp}R = {}^{\perp_{\mathcal{V}}}R$) and Proposition 2.1.23. □

Example 2.1.39. Suppose that in the category $\mathcal{E}$ all intersections of $\mathcal{V}$-monomorphisms exist. Then the pair

$$(StEpi, Mono) = ({}^{\perp_{\mathcal{V}}}\mathcal{M}, \mathcal{M})$$

is an enriched factorisation system in $\mathcal{E}$. When $\mathcal{E}$ is a topos all epimorphisms are strong (and conversely in all categories) so we will write simply $(Epi, Mono)$ for this factorisation system.

Now we extend Lemma 2.1.28 to the enriched setting.

Corollary 2.1.40. *Let Σ be a class of arrows in the underlying category $\mathcal{E}_0$ of a $\mathcal{V}$-category $\mathcal{E}$ which has all tensors and cotensors. Suppose that the ordinary category $\mathcal{E}_0$ is locally presentable. Let $\tilde{\Sigma}$ be the closure of Σ under tensor. Then*

$$(L, R) = (Sat(\tilde{\Sigma} \cup \delta^o(\tilde{\Sigma})), \Sigma^{\perp_\mathcal{V}})$$

is a $\mathcal{V}$-factorisation system on $\mathcal{E}$.

Proof. First we show that (L, R) is a $\mathcal{V}$-prefactorisation system. Firstly we notice that

$$Sat(\tilde{\Sigma} \cup \delta^o\tilde{\Sigma}) = {}^{\perp}(\tilde{\Sigma}^{\perp}) = {}^{\perp}(\Sigma^{\perp_\mathcal{V}}) = {}^{\perp_\mathcal{V}}(\Sigma^{\perp_\mathcal{V}}) \tag{2.3}$$

where the first equality follows from Lemma 2.1.28 the second equality from the dual of Corollary 2.1.35 and the third equality from the fact that $\Sigma^{\perp_\mathcal{V}}$ is closed under cotensor. Now applying $(-)^{\perp_\mathcal{V}}$ to both sides of (2.3) gives us that

$$Sat(\tilde{\Sigma} \cup \delta^o\tilde{\Sigma})^{\perp_\mathcal{V}} = \Sigma^{\perp_\mathcal{V}}$$

as required. The existence of (L, R)-factorisations follows immediately from the equality $\Sigma^{\perp_\mathcal{V}} = \tilde{\Sigma}^{\perp}$ and Lemma 2.1.28. □

Lemma 2.1.41. *Let $(L_\mathcal{C}, R_\mathcal{C})$ and $(L_\mathcal{D}, R_\mathcal{D})$ be $\mathcal{V}$-factorisation systems on $\mathcal{V}$-categories $\mathcal{C}$ and $\mathcal{D}$ respectively which have all tensors and cotensors. Let*

$$\mathcal{C} \underset{U}{\overset{F}{\underset{\perp}{\rightleftarrows}}} \mathcal{D}$$

be an adjunction such that for all $\rho : X \to Y$ in $R_\mathcal{D}$ the arrow $U\rho \in R_\mathcal{C}$. Then for all $l : A \to B$ in $L_\mathcal{C}$ the arrow $Fl \in L_\mathcal{D}$.

Proof. Since $(L_\mathcal{D}, R_\mathcal{D})$ is a $\mathcal{V}$-factorisation system we have that

$$L_\mathcal{D} = {}^{\perp_\mathcal{V}}R_\mathcal{D}$$

and in addition that $\rho^v \in R_\mathcal{D}$ for all $v \in \mathcal{V}$ and $\rho \in R_\mathcal{D}$. Hence by hypothesis $U(\rho^v) \in R_\mathcal{C}$ and $l \perp_\mathcal{V} U(\rho^v)$ for all $v \in \mathcal{V}$ and $\rho \in R_\mathcal{D}$. In particular we have that $l \perp U(\rho^v)$ for all $v \in \mathcal{V}$ and $\rho \in R_\mathcal{D}$. But by Lemma 2.1.22 we have that

$$\forall v \in \mathcal{V}.\ \forall \rho \in R_\mathcal{D}.\ l \perp U(\rho^v) \iff Fl \perp \rho^v$$

and Lemma 2.1.34 tells us that:

$$\forall \rho \in R_\mathcal{D}.\ (\forall v \in \mathcal{V}.\ Fl \perp \rho^v) \iff Fl \perp_\mathcal{V} \rho$$

and so $Fl \in L_\mathcal{D}$ as required. □

2.2 The Jet Factorisation System

In the classical (non-intuitionistic) logic that underpins classical Lie theory it is not possible to form the set of all elements of an n-dimensional Lie group G that are 'infinitesimally close' to the identity element e of G in a rigorous manner. Instead we use the following heuristic: whenever we want to work with elements of G that are infinitesimally close to e we instead work with the formal group law or Lie algebra of G. By contrast the internal logic of a well-adapted model of synthetic differential geometry $\mathcal{E}$ is not Boolean and we have already seen that certain 'infinitesimal' objects (that are not isomorphic to the terminal object) such as

$$D_\infty = \bigcup_{i=1}^{\infty} \{x \in R : x^{k+1} = 0\}$$

exist in $\mathcal{E}$. In this thesis we will exploit these infinitesimals and work directly with the infinitesimal neighbourhood of the identity element. In Section 2.3.2 we will assign to any category $\mathbb{C}$ in a well-adapted model $\mathcal{E}$ a subcategory $\mathbb{C}_\infty$ with the same objects but only those arrows which are infinitesimally close to an identity arrow. The study of this subcategory will correspond to the study of the formal group law of a Lie group. As a first step in the construction of $\mathbb{C}_\infty$ we give a concrete procedure that starts with an arrow f in $\mathcal{E}$ and returns the infinitesimal neighbourhood of the image of f. For example if $f = 0 : 1 \to R$ then we would want the infinitesimal neighbourhood of the image of f to be D_∞ above. Recall from Example 2.1.39 that the $(Epi, Mono)$-factorisation system on $\mathcal{E}$ gives us the image of an arrow f as the mediating object in the $(Epi, Mono)$-factorisation of f:

$$A \twoheadrightarrow im(f) \rightarrowtail B$$

In this section we will define the jet factorisation system on $\mathcal{E}/M$ which will be a slight 'perturbation' of the $(Epi, Mono)$-factorisation system. The mediating object of the factorisation of an arrow f using the jet factorisation system will be the infinitesimal neighbourhood of the image of f that we require.

We will construct the jet factorisation system as an $\mathcal{E}/M$-factorisation system in the slice category $\mathcal{E}/M$. Since both $\mathcal{E}$ and $\mathcal{E}/M$ are Grothendieck toposes the conditions of Corollary 2.1.38 that we use to construct it are easily seen to be satisfied.

2.2.1 Jet Factorisation in the Slice Topos

We define the jet factorisation system on any slice category $\mathcal{E}/M$ of the smooth topos $\mathcal{E}$. Since it is a topos the category $\mathcal{E}$ is locally cartesian closed. Using this fact, we show that for any arrow $f : X \to Y$ in $\mathcal{E}$ both the pullback functor $f^* : \mathcal{E}/Y \to \mathcal{E}/X$ and its left adjoint $\Sigma_f : \mathcal{E}/X \to \mathcal{E}/Y$ preserve the left class of the jet factorisation systems on $\mathcal{E}/X$ and $\mathcal{E}/Y$. This will be used in the next section to define the composition operation on the jet part of a category in $\mathcal{E}$. In the case $M = 1$ the right class of the jet factorisation system has been studied before. They are the formal-etale maps in I.17 of [19] and the formally-open morphisms in Section 1.2 of Volume 3 of [14]. For the standard theory of toposes we refer to [23].

In this section $\mathcal{E}$ will be a smooth topos and M an object of $\mathcal{E}$. To begin with let us recall the definition of slice category. It can be found for example in construction 4 of Section 1.6 in [1].

Definition 2.2.1. The slice category $\mathcal{E}/M$ of a category $\mathcal{E}$ over an object $M \in \mathcal{E}$ has as objects all arrows $f \in \mathcal{E}$ such that the codomain of f is M. To keep track of the domain of f we write the objects of $\mathcal{E}/M$ in the form $(dom(f), f)$. An arrow $g : (X, f) \to (X', f')$ in $\mathcal{E}/M$ is an arrow $g : X \to X'$ in $\mathcal{E}$ such that $f' \circ g = f$, as indicated in

$$\begin{array}{ccc} X & \xrightarrow{g} & X' \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle f'} & \\ & M & \end{array}$$

The following is part of Theorem 1.42 in [13].

Theorem 2.2.2. *Let $\mathcal{E}$ be a topos, X an object of $\mathcal{E}$. Then $\mathcal{E}/X$ is a topos.*

Definition 2.2.3. Let Λ be the following class of arrows in $\mathcal{E}/M$:

$$\Lambda = \{(M, 1_M) \xrightarrow{(1_M,0)} (M \times D, \pi_1) : D \in Spec(Weil)\}$$

Then the jet factorisation system on $\mathcal{E}/M$ is the factorisation system

$$(L_\infty, R_\infty) = ({}^{\perp_\mathcal{V}}(\Lambda^{\perp_\mathcal{V}} \cap \mathcal{M}), \Lambda^{\perp_\mathcal{V}} \cap \mathcal{M})$$

generated by Λ using Corollary 2.1.38 where $\mathcal{V} = \mathcal{E}/M$ and where $\mathcal{M}$ is as in Corollary 2.1.38. Note that since $\mathcal{E}/M$ is a Grothendieck topos all of the conditions in Corollary 2.1.38 are satisfied.

To put this in more concrete terms, an arrow $r : X \to Y$ in $\mathcal{E}/M$ is in the right class R_∞ (and is called jet closed) iff it is an $\mathcal{E}/M$-monomorphism and

$$\begin{array}{ccc} X^{(M\times D,\pi_1)} & \xrightarrow{X^{(1_M,0)}} & X^{(M,1_M)} \\ \downarrow{\scriptstyle r^{(M\times D,\pi_1)}} & & \downarrow{\scriptstyle r^{(M,1_M)}} \\ Y^{(M\times D,\pi_1)} & \xrightarrow{Y^{(1_M,0)}} & Y^{(M,1_M)} \end{array}$$

is a pullback in $\mathcal{E}/M$ for all D in $Spec(Weil)$. An arrow $l : A \to B$ in $\mathcal{E}/M$ is in the left class L_∞ (and is called jet dense) iff for all $r \in R_\infty$

$$\begin{array}{ccc} X^B & \xrightarrow{X^l} & X^A \\ \downarrow{\scriptstyle r^B} & & \downarrow{\scriptstyle r^A} \\ Y^B & \xrightarrow{Y^l} & Y^A \end{array}$$

is a pullback in $\mathcal{E}/M$.

We can relate the jet factorisation systems on different slices over $\mathcal{E}$ by using the fact that $\mathcal{E}$ is locally cartesian closed.

Proposition 2.2.4. *Let $f : G \to M$ be an arrow in $\mathcal{E}$. Let $f^* : \mathcal{E}/M \to \mathcal{E}/G$ be the functor defined by pullback along f. Then f^* preserves exponentials and has both a left adjoint Σ_f and right adjoint Π_f; the left adjoint Σ_f is given by postcomposition with f.*

$$\mathcal{E}/G \underset{\Pi_f}{\overset{\Sigma_f}{\leftarrow f^* -}} \mathcal{E}/M$$

Proof. This is Theorem 2 on page 193 in [23]. ☐

Lemma 2.2.5. *Let $\rho : X \rightarrowtail Y$ be a jet closed arrow in $\mathcal{E}/M$ and $f : G \to M$ an arrow in $\mathcal{E}$. Then $f^*(\rho)$ is a jet-closed arrow in $\mathcal{E}/G$.*

Proof. Since ρ is jet closed in $\mathcal{E}/M$ we have that for all $D \in Spec(Weil)$ the following square is a pullback:

$$\begin{array}{ccc} X^{(M\times D,\pi_1)} & \xrightarrow{X^{(1_M,0)}} & X \\ \downarrow{\scriptstyle \rho^{(M\times D,\pi_1)}} & & \downarrow{\scriptstyle \rho} \\ Y^{(M\times D,\pi_1)} & \xrightarrow{Y^{(1_M,0)}} & Y \end{array}$$

Using the fact that f^* preserves exponentials we see that:

$$f^*\left(\begin{array}{ccc} X^{(M\times D,\pi_1)} & \xrightarrow{X^{(1_M,0)}} & X \\ \big\downarrow{\scriptstyle \rho^{(M\times D,\pi_1)}} & & \big\downarrow{\scriptstyle \rho} \\ Y^{(M\times D,\pi_1)} & \xrightarrow{Y^{(1_M,0)}} & Y \end{array}\right) \cong \begin{array}{ccc} f^*(X)^{(G\times D,\pi_1)} & \xrightarrow{f^*(X)^{(1_G,0)}} & f^*(X) \\ \big\downarrow{\scriptstyle f^*(\rho)^{(G\times D,\pi_1)}} & & \big\downarrow{\scriptstyle f^*(\rho)} \\ f^*(Y)^{(G\times D,\pi_1)} & \xrightarrow{f^*(Y)^{(1_G,0)}} & f^*(Y) \end{array}$$

Then using the fact that f^* is a right adjoint we deduce that the right hand square is a pullback for all $D \in Spec(Weil)$ and so $f^*(\rho)$ is jet-closed in $\mathcal{E}/G$. □

Lemma 2.2.6. *Let $F \dashv U$ be adjoint functors. Suppose that F preserves products. Then:*

$$(UA)^B \cong U(A^{FB})$$

Proof. We will establish a natural bijection between the generalised elements of both sides:

$$\begin{array}{c} X \to (UA)^B \\ \hline X \times B \to UA \\ \hline F(X\times B) \to A \\ \hline FX \to A^{FB} \\ \hline X \to U(A^{FB}) \end{array}$$

as required. □

Lemma 2.2.7. *Let $\rho : X \rightarrowtail Y$ be a jet closed arrow in $\mathcal{E}/G$ and $f : G \to M$ an arrow in $\mathcal{E}$. Then $\Pi_f(\rho)$ is a jet-closed arrow in $\mathcal{E}/M$.*

Proof. Since ρ is jet closed in $\mathcal{E}/G$ we have that for all $D \in Spec(Weil)$ the following square is a pullback:

$$\begin{array}{ccc} X^{(G\times D,\pi_1)} & \xrightarrow{X^{(1_G,0)}} & X \\ \big\downarrow{\scriptstyle \rho^{(G\times D,\pi_1)}} & & \big\downarrow{\scriptstyle \rho} \\ Y^{(G\times D,\pi_1)} & \xrightarrow{Y^{(1_G,0)}} & Y \end{array}$$

Using Lemma 2.2.6 we see that:

$$\Pi_f\left(\begin{array}{ccc} X^{(G\times D,\pi_1)} & \xrightarrow{X^{(1_G,0)}} & X \\ \big\downarrow{\scriptstyle \rho^{(G\times D,\pi_1)}} & & \big\downarrow{\scriptstyle \rho} \\ Y^{(G\times D,\pi_1)} & \xrightarrow{Y^{(1_G,0)}} & Y \end{array}\right) \cong \begin{array}{ccc} \Pi_f(X)^{(M\times D,\pi_1)} & \xrightarrow{\Pi_f(X)^{(1_M,0)}} & \Pi_f(X) \\ \big\downarrow{\scriptstyle \Pi_f(\rho)^{(M\times D,\pi_1)}} & & \big\downarrow{\scriptstyle \Pi_f(\rho)} \\ \Pi_f(Y)^{(M\times D,\pi_1)} & \xrightarrow{\Pi_f(Y)^{(1_M,0)}} & \Pi_f(Y) \end{array}$$

Then using the fact that Π_f is a right adjoint we deduce that the right hand square is a pullback for all $D \in Spec(Weil)$ and so $\Pi_f(\rho)$ is jet-closed in $\mathcal{E}/M$. □

Corollary 2.2.8. *Let l be jet dense in $\mathcal{E}/G$ and $f : G \to M$ an arrow in $\mathcal{E}$. Then $\Sigma_f(l)$ is jet dense in $\mathcal{E}/M$.*

Proof. Follows immediately from Lemma 2.2.5 and Lemma 2.1.41. □

Corollary 2.2.9. *Let λ be a jet dense arrow in $\mathcal{E}/M$ and $f : G \to M$ an arrow in $\mathcal{E}$. Then $f^*(\lambda)$ is jet dense in $\mathcal{E}/G$.*

Proof. Follows immediately from Lemma 2.2.7 and Lemma 2.1.41 □

2.2.2 Jet Factorisation Using Neighbours

The jet factorisation system presented in Section 2.2.1 can be thought of as a 'perturbation' of the standard $(Epi, Mono)$-factorisation in Example 2.1.39. Intuitively speaking, if $f : A \to B$ is a jet dense arrow and b is an element of B then although there might not exist an element a of A such that $fa = b$ there does exist an element a' of A such that fa' is 'infinitesimally close' to b. We can give a similar heuristic description for the jet closed arrows. If $g : X \rightarrowtail Y$ is a jet closed arrow then it is a monomorphism by definition. But g satisfies an additional condition: if x is an element of X and y is an element of Y such that gx is infinitesimally close to y then there exists an element x' in X such that $gx' = y$. In this section we make these ideas precise by defining a reflexive relation $\sim$ in the internal logic of the topos $\mathcal{E}/M$ for which $a \sim b$ encodes the idea that b is contained in some infinitesimal perturbation (or jet) which is based at a. Then we define a factorisation system using this relation which corresponds to our intuitive idea of perturbing the $(Epi, Mono)$-factorisation in $\mathcal{E}/M$. Finally we show that this factorisation system in fact coincides with the jet factorisation system.

First we recall the definition of generalised element in a category from Definition 1.1 in Part II of [19].

Definition 2.2.10. Let R be an object in a category $\mathcal{E}$. An element of R is an arrow in $\mathcal{E}$ with codomain R. The domain of the arrow is called the stage of definition of the element.

Notation 2.2.11. We write $r \in_X R$ to denote that r is an arrow $X \to R$ in $\mathcal{E}$ and hence r is an element of R at stage of definition X. When we work with an arbitrary fixed stage of definition we will sometimes write simply $r \in R$ where it causes no confusion. For interpreting existential quantification and disjunction we will need to consider covers $(\iota_i : X_i \to X)_i$ of the stage of definition X. Then if $a \in_X R$ will write $a|_{X_i}$ for the element $a\iota_i \in_{X_i} R$.

Let D_W be a Weil spectrum in $\mathcal{E}$. Then we abuse notation by writing D_W for the object $(M \times D_W, \pi_1)$ of $\mathcal{E}/M$.

Definition 2.2.12. Let $a, b \in_X B$ where X and B are objects of the topos $\mathcal{E}/M$. Then $a \sim b$ iff the proposition

$$\bigvee_{W \in Weil} \exists \phi \in B^{D_W}.\ \exists d \in D_W.\ \phi(0) = a \wedge \phi(d) = b$$

holds in the internal logic of $\mathcal{E}/M$.

Explicitly: there exists a cover $(\iota_i : X_i \to X)_{i \in I}$ in $\mathcal{E}/M$ such that for each i there exists an object $D_{W_i} \in Spec(Weil)$, an arrow $\phi_i : X_i \times D_{W_i} \to B$ and an arrow $d_i : X_i \to D_{W_i}$ such that

$$\begin{array}{ccc} X_i & \xrightarrow{(1_{X_i},0)} & X_i \times D_{W_i} \\ \big\downarrow{\scriptstyle a|_{X_i}} & & \big\downarrow{\scriptstyle \phi_i} \\ B & \xrightarrow{\quad 1_B \quad} & B \end{array}$$

and

$$\begin{array}{ccc} X_i & \xrightarrow{(1_{X_i},d_i)} & X_i \times D_{W_i} \\ \big\downarrow{\scriptstyle b|_{X_i}} & & \big\downarrow{\scriptstyle \phi_i} \\ B & \xrightarrow{\quad 1_B \quad} & B \end{array}$$

commute.

Remark 2.2.13. The relation $\sim$ is not always symmetric. In fact it is not symmetric in the case $B = D$ and $M = 1$ as described in Example 2.3.21.

Definition 2.2.14. The relation $\approx$ is the transitive closure of $\sim$ in the internal logic of $\mathcal{E}/M$. This means that for $a, b \in B$ we have $a \approx b$ iff the proposition

$$\bigvee_{n\in\mathbb{N}} \exists \vec{x} \in B^n. \bigwedge_{1\leq k\leq n-1} (\pi_k\vec{x} \sim \pi_{k+1}\vec{x}) \wedge (\pi_1\vec{x} = a) \wedge (\pi_n\vec{x} = b)$$

holds in the internal logic of $\mathcal{E}/M$.

In terms of covers: let $a, b \in_X B$ where X and B are objects of $\mathcal{E}/M$. Then $a \approx b$ iff there exists a cover $(\iota_i : X_i \to X)_{i\in I}$ and for each i there exists a natural number n_i and elements $x_{i0}, x_{i1}, ..., x_{in_i} \in_{X_i} B$ such that

$$a|_{X_i} = x_{i0} \sim x_{i1} \sim ... \sim x_{in_i} = b|_{X_i}$$

Remark 2.2.15. For any arrow $f : A \to B$ we have that $a \sim a'$ in A implies that $fa \sim fa'$ in B. Indeed if we have $D \in \mathit{Spec}(Weil)$, $\phi \in B^D$ and $d \in D$ such that $f(0) = a$ and $f(d) = a'$ then for the same D and d we see that $\psi = B^f \circ \phi$ has $\psi(0) = fa$ and $\psi(d) = fa'$.

We can easily iterate this procedure to obtain that $a \approx a'$ in A implies $fa \approx fa'$ in B.

Definition 2.2.16. Let $f : A \to B$ be an arrow in $\mathcal{E}/M$. Then f is $\mathcal{W}$-dense (or $f \in L_{\mathcal{W}}$) iff the proposition

$$\forall b \in B.\ \exists a \in A.\ fa \approx b$$

holds in the internal logic of $\mathcal{E}/M$.

Explicitly: for all $b \in_X B$ there exists a cover $(\iota_i : X_i \to X)_{i\in I}$ and elements $a_i \in_{X_i} A$ such that $f(a_i) \approx b|_{X_i}$.

Definition 2.2.17. Let $g : A \to B$ be an arrow in $\mathcal{E}/M$. Then g is $\mathcal{W}$-closed (or $g \in R_{\mathcal{W}}$) iff the propositions

$$\forall a \in A.\ \forall b \in B.\ ga \approx b \implies (\exists c \in A.\ gc = b)$$

and

$$\forall a, a' \in A.\ ga = ga' \implies (a \approx a')$$

hold in the internal logic of $\mathcal{E}/M$.

Explicitly the first condition is: for all $a \in_X A$ and $b \in_X B$ such that $ga \approx b$ there exists a cover $(\iota_i : X_i \to X)_{i\in I}$ and elements $c_i \in_{X_i} A$ such that $a|_{X_i} \approx c_i$

and $gc_i = b|_{X_i}$. Since the second condition only uses universal quantification and conjunction it is not necessary to pass to a cover.

Remark 2.2.18. Note that in the sequel the right class of the jet factorisation system will turn out not to be simply $R_{\mathcal{W}}$ but its intersection with the monomorphisms in $\mathcal{E}/M$. The larger class $R_{\mathcal{W}}$ will be useful in Section 2.2.3.

From now on we will work entirely in the internal logic of $\mathcal{E}/M$. The interested reader is welcome to translate the statements below into their external versions involving covers by applying the sheaf semantics explained in Section VI.7 of [23].

Lemma 2.2.19. *Let $g : B \rightarrowtail E$ be a $\mathcal{W}$-closed monomorphism. Suppose that $gb \sim gb'$ in E. Then $b \sim b'$ in B.*

Proof. Since $gb \sim gb'$ there exists $D \in Spec(Weil)$, $\phi \in E^D$ and $d \in D$ such that $\phi(0) = gb$ and $\phi(d) = gb'$. However it is immediate from the fact that g is $\mathcal{W}$-closed that ϕ is in the image of $g^D : B^D \rightarrowtail E^D$ and so there exists ψ such that $\phi = g^D\psi$. But $g(\psi(0)) = gb$ and $g(\psi(d)) = gb'$ hence $\psi(0) = b$ and $\psi(d) = b'$ and $b \sim b'$ as required. □

Corollary 2.2.20. *Let $g : B \rightarrowtail E$ be a $\mathcal{W}$-closed monomorphism. Suppose that $gb \approx gb'$ in E. Then $b \approx b'$ in B.*

Proof. Let $gb = e_0 \sim e_1 \sim ... \sim e_n = gb'$ exhibit $gb \approx gb'$. Then the fact that g is $\mathcal{W}$-closed combined with $e_0 = gb$ implies that there exists $b_1 \in B$ such that $e_1 = gb_1$. Then by Lemma 2.2.19 we see that $b \sim b_1$. The result follows easily by iterating this procedure. □

Lemma 2.2.21. *Let $h : A \to E$ be an arrow in $\mathcal{E}/M$. Then there exist $g \in R_{\mathcal{W}}$ and $f \in L_{\mathcal{W}}$ such that g is a monomorphism and $h = gf$. The mediating object in the factorisation has the presentation*

$$B = \{x \in E : \exists a \in A.\ ha \approx x\} \xrightarrow{g} E$$

in the internal logic of $\mathcal{E}/M$.

Proof. It is immediate that h factors through the subobject B because the relation $\approx$ is reflexive. Write $h = gf$ for this factorisation.

To see that g is $\mathcal{W}$-closed let $b \in B$ and $e \in E$ such that $gb \approx e$. By the definition of B there exists an $a \in A$ such that $ha \approx gb$. Hence by the transitivity of $\approx$ we obtain that $ha \approx e$. So e lies in the subobject B and so g is $\mathcal{W}$-closed as required.

To see that f is $\mathcal{W}$-dense let $b \in B$. Now by the definition of B there exists an $a \in A$ such that $ha \approx gb$. But since g is a $\mathcal{W}$-closed monomorphism we can use Lemma 2.2.20 we deduce that $fa \approx b$ as required. □

Proposition 2.2.22. *Let $\mathcal{M}$ be the class of monomorphisms in $\mathcal{E}/M$. Then the pair*

$$(L, R) = (L_{\mathcal{W}}, R_{\mathcal{W}} \cap \mathcal{M})$$

defines a $(\mathcal{E}/M)$-factorisation system.

Proof. We will check the conditions of Lemma 2.1.9. The existence of factorisations is Lemma 2.2.21 and it is clear that the classes $L_{\mathcal{W}}$ and $R_{\mathcal{W}} \cap \mathcal{M}$ are replete.

It remains to show that for all $\mathcal{W}$-closed monomorphisms $g : C \rightarrowtail E$ and all $\mathcal{W}$-dense arrows $f : A \to B$ we have that $f \perp_{\mathcal{E}/M} g$. That means we need to show that the square

$$\begin{array}{ccc} C^B & \xrightarrow{C^f} & C^A \\ \big\downarrow{\scriptstyle g^B} & & \big\downarrow{\scriptstyle g^A} \\ E^B & \xrightarrow{E^f} & E^A \end{array}$$

is a pullback. So suppose that $\phi \in E^B$ and $\psi \in C^A$ such that $\phi f = g\psi$. We define $\xi \in C^B$ as follows. Start with $b \in B$. Since f is $\mathcal{W}$-dense there exists $a \in A$ such that $fa \approx b$. Then by Remark 2.2.15 we have that $g\psi a = \phi f a \approx \phi b$. Now since g is $\mathcal{W}$-closed we have that there exists $c \in C$ such that $gc = \phi b$. This c is unique because g is monic. So finally we define $\xi b = c$. It is immediate that $g\xi b = gc = \phi b$. From the equation $g\xi f a = \phi f a = g\psi a$ we deduce that $\xi f a = \psi a$ as required. □

Proposition 2.2.23. *Let $f : A \rightarrowtail B$ be a monomorphism in $\mathcal{E}/M$. Then f*

is $\mathcal{W}$-closed iff for all $D \in Spec(Weil)$ the square

$$\begin{array}{ccc} A^{(M\times D,\pi_1)} & \xrightarrow{A^0} & A \\ \downarrow f^D & & \downarrow f \\ B^{(M\times D,\pi_1)} & \xrightarrow{B^0} & B \end{array} \tag{2.4}$$

is a pullback.

Proof. We will show that $L_\infty \subset L_\mathcal{W}$ and $R_\infty \subset R_\mathcal{W}$. This will suffice to prove the result because

$$L_\infty \subset L_\mathcal{W} \implies L_\mathcal{W}^\perp \subset L_\infty^\perp \implies R_\mathcal{W} \subset R_\infty$$

To show that $L_\infty \subset L_\mathcal{W}$ we need to show that for all $D \in Spec(Weil)$ the arrow $(M, 1_M) \to (M \times D, \pi_1)$ is in $L_\mathcal{W}$. For this it will suffice to show that for all $b \in (M \times D, \pi_1)$ we have $0 \approx b$. Here 0 denotes the global element $(1_M, 0) : (M, 1_M) \to (M \times D, \pi_1)$. So we choose $D_W = (M \times D, \pi_1)$, $\phi = 1_{M\times D}$ and $d = b$. Then $\phi(0) = 0$ and $\phi(d) = b$.

To show that $R_\infty \subset R_\mathcal{W}$ let f be a monomorphism, let $a \in A$ and $b \in B$ such that $fa \sim b$ and suppose that the square in (2.4) is a pullback. The condition $fa \sim b$ means that there is a $D_W \in Spec(Weil)$, a $\phi \in B^{(M\times D_W,\pi_1)}$ and a $d \in (M \times D_W, \pi_1)$ such that $\phi(0) = fa$ and $\phi(d) = b$. Since $\phi(0) = fa$ we can induce a $\psi \in A^{(M\times D,\pi_1)}$ using the pair (a, ϕ). But then we have $f\psi(d) = \phi(d) = b$.

We now iterate this argument to obtain that f is $\mathcal{W}$-closed as required. □

Corollary 2.2.24. *The $(L_\mathcal{W}, R_\mathcal{W} \cap \mathcal{M})$ factorisation system and the jet factorisation system coincide in $\mathcal{E}/M$.*

2.2.3 Stability Properties of the Jet Factorisation

In this section and the next we draw further analogies between the jet factorisation system on $\mathcal{E}$ and the $(Epi, Mono)$-factorisation system. Recall that for all factorisation systems the left class is closed under colimits and the right class is closed under limits. The $(Epi, Mono)$-factorisation system has the additional property that the left class is closed under pullback and the right class is closed under pushouts. We first prove that the right class of the jet factorisation is closed under pushout using a standard result concerning the stability of colimits under pullback in a topos. Then we identify a condition on an arrow g in the left class of the jet factorisation system which guarantees that the pullback of g along a $\mathcal{W}$-closed arrow k is again jet dense.

Recall from Proposition 2.2.4 that for all $f : X \to Y$ the functor f^* defined by pullback along f preserves colimits. Recall also from Proposition 1.3.8 that for any $D \in Spec(Weil)$ the functor $(-)^D$ has a right adjoint.

Proposition 2.2.25. *In a topos, consider the pushout*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle k} \\ C & \xrightarrow{h} & C \end{array}$$

with g a monomorphism. Then k is a monomorphism as well and the square is also a pullback.

Proof. This is Proposition 5.9.10 in [2]. □

Proposition 2.2.26. *Let $r_1 : X_1 \rightarrowtail Y_1$ be a jet closed arrow in $\mathcal{E}/M$ and*

$$\begin{array}{ccc} X_1 & \xrightarrow{f} & X_2 \\ {\scriptstyle r_1}\downarrow & & \downarrow{\scriptstyle r_2} \\ Y_1 & \xrightarrow{g} & Y_2 \end{array}$$

be a pushout in $\mathcal{E}$. Then the arrow r_2 is jet closed.

Proof. Since $Y_2 \cong Y_1 +_{X_1} X_2$ the conclusion of Proposition 1.3.8 tells us that $Y_2^D \cong Y_1^D +_{X_1^D} X_2^D$. This means that

$$\begin{array}{ccc} (X_1^D, Y_2^0 g^D r_1^D) & \xrightarrow{f^D} & (X_2^D, Y_2^0 r_2^D) \\ {\scriptstyle r_1^D}\downarrow & & \downarrow{\scriptstyle r_2} \\ (Y_1^D, Y_2^0 g^D) & \xrightarrow{g^D} & (Y_2^D, Y_2^0) \end{array}$$

is a pushout in $(\mathcal{E}/M)/Y_2$ and so

$$\begin{array}{ccc} (X_1^D \times_{Y_2} X_2, Y_2^0 g^D r_1^D \pi_1) & \xrightarrow{f^D \times_{Y_2} X_2} & (X_2^D \times_{Y_2} X_2, Y_2^0 r_2^D \pi_1) \\ {\scriptstyle r_1^D \times_{Y_2} X_2}\downarrow & & \downarrow{\scriptstyle r_2^D \times_{Y_2} X_2} \\ (Y_1^D \times_{Y_2} X_2, Y_2^0 g^D \pi_1) & \xrightarrow{g^D \times_{Y_2} X_2} & (Y_2^D \times_{Y_2} X_2, Y_2^0 \pi_1) \end{array}$$

is a pushout in $(\mathcal{E}/M)/X_2$ because the functor r_2^* defined by pulling back along r_2 has a right adjoint by Proposition 2.2.4. Therefore

$$\begin{CD}
X_1^D \times_{Y_2} X_2 @>{f^D \times_{Y_2} X_2}>> X_2^D \times_{Y_2} X_2 \\
@V{r_1^D \times_{Y_2} X_2}VV @VV{r_2^D \times_{Y_2} X_2}V \\
Y_1^D \times_{Y_2} X_2 @>{g^D \times_{Y_2} X_2}>> Y_2^D \times_{Y_2} X_2
\end{CD}$$

is a pushout in $\mathcal{E}/M$. We now find simpler descriptions of the objects involved in this pushout square. In the diagram

$$\begin{CD}
Y_1^D \times_{Y_2} X_2 @>>> X_1 @>{f}>> X_2 \\
@VVV @VV{r_1}V @VV{r_2}V \\
Y_1^D @>{Y_1^0}>> Y_1 @>{f'}>> Y_2
\end{CD}$$

the right hand square is a pullback by Proposition 2.2.25 and the composite rectangle is a pullback by definition. Hence $Y_1^D \times_{Y_2} X_2 \cong X_1^D$ because r_1 is jet closed. Similarly in the diagram

$$\begin{CD}
X_2^D \times_{Y_2} X_2 @>>> X_2 @>{1_{X_2}}>> X_2 \\
@VVV @VV{1_{X_2}}V @VV{r_2}V \\
X_2^D @>{X_2^0}>> X_2 @>{r_2}>> Y_2
\end{CD}$$

the right hand square is a pullback because r_2 is a monomorphism and the whole rectangle is a pullback by definition. Hence $X_2^D \times_{Y_2} X_2 \cong X_2^D$. Finally, in the diagram

$$\begin{CD}
X_1^D \times_{Y_2} X_2 @>>> X_1^D @>{X_1^0}>> X_1 @>{f}>> X_2 \\
@VVV @VV{r_1^D}V @VV{r_1}V @VV{r_2}V \\
X_1^D @>{r_1^D}>> Y_1^D @>{Y_1^0}>> Y_1 @>{f'}>> Y_2
\end{CD}$$

we have already seen that the two squares to the right are pullbacks and the outer rectangle is a pullback by definition. In addition r_1^D is a monomorphism because $(-)^D$ preserves monomorphisms. Hence $X_1^D \times_{Y_2} X_2 \cong X_1^D$ and $Q \cong X_2^D +_{X_1^D} X_1^D \cong X_2^D$ as required. □

Now we turn to the special case when jet dense arrows are stable under pullback.

Proposition 2.2.27. *Let g be jet dense and k be $\mathcal{W}$-closed in $\mathcal{E}/M$. Suppose that the relation $\approx$ is symmetric on the object E and that the square*

$$\begin{array}{ccc} A & \xrightarrow{h} & B \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ C & \xrightarrow{k} & E \end{array}$$

is a pullback. Then f is also jet dense.

Proof. Recall that an arrow in $\mathcal{E}/M$ is jet dense iff it is $\mathcal{W}$-dense. Let $c \in C$. We need to show that there exists $a \in A$ such that $fa \approx c$. Since g is $\mathcal{W}$-dense there exists $b \in B$ such that $gb \approx kc$. Since $\approx$ is symmetric on E we see that also $kc \approx gb$. Now k is $\mathcal{W}$-closed so there exists $c' \in C$ such that $c \approx c'$ and $kc' = gb$. The $a \in A$ that we require is the one defined by the pair $a = (c', b)$.

We now confirm that $f(c', b) = c' \approx c$. First we see that $kc' = gb \approx kc$ and so there exists $c'' \in C$ such that $c' \approx c''$ and $kc'' = kc$. But now by the definition of $\mathcal{W}$-closed we have that $c'' \approx c$ and by transitivity of $\approx$ that $c' \approx c$ as required. □

Corollary 2.2.28. *Let g be jet dense and k be jet closed. Suppose that the relation $\approx$ is symmetric on E and the square*

$$\begin{array}{ccc} A & \overset{h}{\rightarrowtail} & B \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ C & \overset{k}{\rightarrowtail} & E \end{array}$$

is a pullback. Then f is also jet dense.

2.2.4 Pullback Stability in the Cahiers Topos

If we consider the proof of Proposition 2.2.27 the condition that k is jet closed is used to ensure that we obtain a genuine element of the pullback A: we needed to find $c' \in C$ and $b \in B$ such that $kc' = gb$ on the nose rather than just up to an infinitesimal perturbation. But intuitively speaking it seems as though every generalised element of a space should be expressible as a perturbation of some 'base' or 'reduced' element that has no infinitesimal variation. To make this idea precise we will choose a specific model of synthetic differential geometry: the Cahiers topos $\mathcal{E}_W$ of Definition 1.2.10. Recall that the representable objects of $\mathcal{E}_W$ are all of the form $D_W \times M$ for some manifold M and $D_W \in Spec(Weil)$. This will allows us to appropriately define the base element of a generalised element. We then show that every generalised element is an infinitesimal perturbation of its base and use this to prove that in this particular topos the pullback of any jet dense arrow along an arbitrary arrow is again jet dense.

Definition 2.2.29. Let $a : X \to B$ in $\mathcal{E}$ where X is representable. Therefore we can write $X = M \times D_{W'}$ for some manifold M and some $D_{W'} \in Spec(Weil)$. Then the base $\overline{a}$ of a is the arrow $a \circ 0 \circ ! : X \to B$:

$$\overline{a} = M \times D_{W'} \xrightarrow{M\times !} M \times 1 \xrightarrow{M\times 0} M \times D_{W'} \xrightarrow{a} B$$

Lemma 2.2.30. *Let $a : X \to B$ in $\mathcal{E}$ where $X = M \times D_{W'}$ is representable. Then we have $\overline{a} \sim a$.*

Proof. We choose the singleton identity cover and $D_W = D_{W'}$. For ϕ and d we choose:

$$\phi = M \times D_W \xrightarrow{\pi_1} M \xrightarrow{\check{b}} B^{D_W}$$

and

$$d = M \times D_W \xrightarrow{\pi_2} D_W$$

therefore $\phi(d) = b$ and $\phi(0) = \overline{b}$. □

Lemma 2.2.31. *Let $a \sim b$. Then we have the equality $\overline{a} = \overline{b}$.*

Proof. Suppose that $(\iota_i : X_i \to X)_{i\in I}$, D_{W_i}, ϕ_i and d_i witness that $a \sim b$ for $X = M \times D_{W'}$ representable. By the definition of the Cahiers coverage we can write $X_i = M_i \times D_{W'}$ for some M_i that cover M when interpreted as manifolds. Then

$$\begin{aligned}
\overline{b}|_{X_i} &= M_i \times D_{W'} \xrightarrow{\iota_i \times D_{W'}} M \times D_{W'} \xrightarrow{M_i\times !} M_i \xrightarrow{M\times 0} M \times D_{W'} \xrightarrow{b} B \\
&= M_i \times D_{W'} \xrightarrow{M_i\times !} M_i \times 1 \xrightarrow{M_i\times 0} M_i \times D_{W'} \xrightarrow{\iota_i \times D_{W'}} M \times D_{W'} \xrightarrow{b} B \\
&= M_i \times D_{W'} \xrightarrow{M_i\times !} M_i \times 1 \xrightarrow{M_i\times 0} M_i \times D_{W'} \xrightarrow{b|_{X_i}} B \\
&= M_i \times D_{W'} \xrightarrow{M_i\times !} M_i \times 1 \xrightarrow{M_i\times 0} M_i \times D_{W'} \xrightarrow{(\phi_i, d_i)} B^{D_{W_i}} \times D_{W_i} \xrightarrow{ev} B \\
&= M_i \times D_{W'} \xrightarrow{M_i\times !} M_i \times 1 \xrightarrow{M_i\times 0} M_i \times D_{W'} \xrightarrow{(\phi_i, 0)} B^{D_{W_i}} \times D_{W_i} \xrightarrow{ev} B \\
&= \overline{a}|_{X_i}
\end{aligned}$$

where the fifth equality is due to the fact that there is only one arrow $0 : N \to D$ for any manifold N and any $D \in Spec(Weil)$. The result follows because the ι_i are jointly epimorphic. □

Corollary 2.2.32. *Let $a \approx b$. Then $\overline{a} = \overline{b}$.*

Proposition 2.2.33. *Let $g : A \to B$ be $\mathcal{W}$-dense in $\mathcal{E}$ and*

$$\begin{array}{ccc} P & \xrightarrow{\pi_2} & A \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle g} \\ C & \xrightarrow{k} & B \end{array}$$

be a pullback in $\mathcal{E}$. Then the arrow π_1 is jet dense.

Proof. Let $c \in_X C$. Then $kc \in_X B$. Since g is jet dense there exists a cover $(\iota_i : X_i \rightarrowtail X)_{i \in I}$ and elements $a_i \in_{X_i} A$ such that $g(a_i) \approx (kc)|_{X_i}$. By Corollary 2.2.32 we have that

$$g\overline{a_i} = \overline{g(a_i)} = \overline{k(c|_{X_i})} = k\left(\overline{c|_{X_i}}\right)$$

because pre- and postcomposition commute. So we have that

$$(\overline{c|_{X_i}}, \overline{a_i}) \in_{X_i} P$$

but $\pi_1(\overline{c|_{X_i}}, \overline{a_i}) = \overline{c|_{X_i}} \sim c|_{X_i}$ hence π_1 is jet dense as required. □

2.3 The Basic Adjunction

In the final two sections of this Chapter we will construct the adjunction which underpins our treatment of Lie theory. The domain of the left adjoint will be a full subcategory $Cat_\infty(\mathcal{E})$ of the category $Cat(\mathcal{E})$ of internal categories in $\mathcal{E}$ whose objects will be called jet categories. The role played by jet categories in this thesis will be roughly analogous to the role played by formal group laws in classical Lie theory and as such they will only contain infinitesimal data in the arrow space. Intuitively $Cat_\infty(\mathcal{E})$ consists of the categories such that every arrow is infinitesimally close to an identity arrow. In Section 2.3.2 we make this idea precise using the jet factorisation system on the slice categories $\mathcal{E}/M$ introduced by Definition 2.2.3. We will define by hand a functor $(-)_\infty$

that assigns to each category $\mathbb{C}$ a jet category $\mathbb{C}_\infty$ and show that $(-)_\infty$ is a coreflection

$$Cat_\infty(\mathcal{E}) \underset{(-)_\infty}{\overset{\iota}{\rightleftarrows}} Cat(\mathcal{E}) \quad (\perp)$$

where ι is the full inclusion.

The domain of the right adjoint of our main adjunction will be a full subcategory $Cat_{int}(\mathcal{E})$ of $Cat(\mathcal{E})$ whose objects will be called integral complete categories. The role played by integral complete categories will be analogous to the role played by Lie groupoids in the classical theory. Intuitively $Cat_{int}(\mathcal{E})$ consists of the categories in which it is possible to integrate certain infinitesimal data into macroscopic arrows. This choice will be justified in Chapter 4 where it will turn out to be the property that we require to prove Lie's second theorem in this context. In Section 2.3.3 we make this precise by introducing the integral factorisation system on $Cat(\mathcal{E})$ in Definition 2.3.29. We will use the established theory relating reflective subcategories and factorisation systems to obtain a functor $(-)_{int}$ which assigns to each category $\mathbb{C}$ an integral complete category $\mathbb{C}_{int}$ which is a reflection

$$Cat(\mathcal{E}) \underset{j}{\overset{(-)_{int}}{\rightleftarrows}} Cat_{int}(\mathcal{E}) \quad (\perp)$$

where j is the full inclusion. The composite of the coreflection $\iota \dashv (-)_\infty$ and the reflection $(-)_{int} \dashv j$ gives us the basic adjunction that we require:

$$Cat_\infty(\mathcal{E}) \underset{(-)_\infty}{\overset{(-)_{int}}{\rightleftarrows}} Cat_{int}(\mathcal{E}) \quad (\perp)$$

To construct the integral factorisation system we will need two categories which are naturally constructed from the unit interval I that is an object of all of our well-adapted models. In Chapter 3 we will need to produce some very similar constructions so we will now give some general procedures to create all the different types of category that we require from a space X.

2.3.1 Categories from Spaces

In the sequel we will require a groupoid ∇I and a category $\mathbb{I}$ to be the appropriate 'representing object for paths' in $Gpd(\mathcal{E})$ and $Cat(\mathcal{E})$ respectively. That is to say ∇I and $\mathbb{I}$ will play an analogous role to that played by the unit interval I in the category Man. In this section we construct these two categories. In addition for every object X in $\mathcal{E}$ we construct a category $\mathbb{N}_X$ which has base space X and, intuitively speaking, has an (unique) arrow $x \to y$ iff y is infinitesimally close to x.

Definition 2.3.1. Let C, M, s and t denote the arrow space, the object space, the source map and the target map respectively of a category $\mathbb{C}$ in $\mathcal{E}$. Then $\mathbb{C}$ is a preorder iff the arrow $(s,t) : C \to M \times M$ is a monomorphism.

Definition 2.3.2. The functor $\mathbb{N}_- : \mathcal{E} \to Cat(\mathcal{E})$ is defined as follows. The category $\mathbb{N}_X$ is the preorder with underlying reflexive graph

$$\{(a,b) \in X^2 : a \approx b\} \; \overset{\pi_1}{\underset{\pi_2}{\overset{\longrightarrow}{\underset{\longrightarrow}{\xleftarrow{\;\Delta\;}}}}} \; X$$

where Δ denotes the diagonal map. The composition of $\mathbb{N}_X$ is uniquely determined by this data because (π_1, π_2) is a monomorphism. Given an arrow $f : X \to Y$ in $\mathcal{E}$ the internal functor $\mathbb{N}f$ is given by the pair $(f \times f, f)$. This is always a reflexive graph homomorphism because

$$a \approx b \implies fa \approx fb$$

by Remark 2.2.15 and it is easily seen to preserve composition.

Example 2.3.3. We formulate an alternative description of the category $\mathbb{N}_I$. It is easy to see that $a \approx_I a + d$ for all $a \in I$ and $d \in D_\infty$ and so we have an arrow $I \times D_\infty \to \mathbb{N}_I^{\mathbf{2}}$ defined by $(a,d) \mapsto (a, a+d)$. We claim that this arrow is invertible, for which it will suffice to show that the arrow $\mathbb{N}_I^{\mathbf{2}} \to I^2$ defined by $(a,b) \mapsto (a, b-a)$ factors through $I \times D_\infty \rightarrowtail I^2$. To do this we need to show that for all $(a,b) \in I^2$ if $a \approx b$ then $b - a$ is nilpotent. By an easy induction we see that it will suffice to show that the corresponding property with $\sim$ in place of $\approx$ holds. (The induction step uses the fact that if $(b-a)^k = 0$ and $(c-b)^k = 0$ then $(c-a)^{n+k} = 0$.) So suppose that $a \sim b$. This means that there exists an object $D_W \in Spec(Weil)$, and element $\phi \in I^{D_W}$ and an element $d \in D_W$ such that $\phi(0) = a$ and $\phi(d) = b$. But by the Kock-Lawvere axiom (and the fact that the inclusion $I \rightarrowtail R$ is jet closed) we have that

$$b = a + N$$

for some nilpotent N as required. This means that we can give the following alternative description of the category $\mathbb{N}_I$.

The category $\mathbb{N}_I$ has underlying reflexive graph isomorphic to

$$I \times D_\infty \underset{\pi_1}{\overset{+}{\underset{\longleftarrow e \longrightarrow}{\rightrightarrows}}} I$$

where $e = (1_I, 0)$ and composition $I \times D_\infty \times D_\infty \to I \times D_\infty$ defined by

$$(a, d, d') \mapsto (a, d + d')$$

Similarly we have that $\mathbb{N}^{\mathbf{2}}_{I^k} \cong I^k \times D^k_\infty$ and so $\mathbb{N}_{I^k} \cong (\mathbb{N}_I)^k$ as categories.

Definition 2.3.4. The functor $\nabla : \mathcal{E} \to Gpd(\mathcal{E})$ is defined as follows. The groupoid ∇X has underlying reflexive graph

$$X \times X \underset{\pi_2}{\overset{\pi_1}{\underset{\longleftarrow \Delta \longrightarrow}{\rightrightarrows}}} X$$

and again the composition of ∇X is uniquely determined by this data. The inverse $i_{\nabla X} : X^2 \to X^2$ is defined by $(a, b) \mapsto (b, a)$. Given an arrow $f : X \to Y$ in $\mathcal{E}$ the internal groupoid homomorphism ∇f is given by the pair $(f \times f, f)$.

Definition 2.3.5. The category $\mathbb{I}$ in $\mathcal{E}$ is defined as follows. Let J be the set of all smooth functions vanishing on the closed subset

$$\{(a, b) : a \leq b\} \subset \mathbb{R}^2$$

This gives an object $[2, J]$ in the category $\mathcal{C}$ of Definition 1.1.1, which is clearly point determined and so lies in each of $\mathcal{C}_W$, $\mathcal{C}_{jet}$, $\mathcal{C}_{fp}$ and $\mathcal{C}_{germ}$. In each case let L be the corresponding representable object in the corresponding smooth topos $\mathcal{E}$. Then $L \rightarrowtail I \times I$ is a monomorphism in $\mathcal{E}$. The underlying graph of $\mathbb{I}$ is

$$L \underset{\pi_2}{\overset{\pi_1}{\underset{\longleftarrow \Delta \longrightarrow}{\rightrightarrows}}} I$$

and the composition of $\mathbb{I}$ is uniquely determined by this data.

Remark 2.3.6. Since every function $f \in J$ is flat on the subset $\{(a,b) : a \leq b\}$ we see that the subobject $L \rightarrowtail I \times I$ is jet closed.

2.3.2 The Jet Part of a Category

We define the (asymmetric) jet part of a category in a smooth topos $\mathcal{E}$. Intuitively the arrow space of the jet part will consist of all the elements of the category which we can reach along an infinitesimally small source constant path starting at an identity arrow. We can put the structure of a reflexive graph on these arrows as follows.

Notation 2.3.7. In this section $\mathbb{C}$ will denote a category in $\mathcal{E}$ with underlying reflexive graph

$$\mathbb{C} = \left(C \;\underset{t}{\overset{s}{\rightrightarrows}}\!\!\!\!\!\!\overset{}{\underset{}{\xleftarrow{\;e\;}}}\; M \right)$$

and composition μ. We will write $n_\times C = C \;{}_t\!\times_s C \;{}_t\!\times_s \cdots {}_t\!\times_s C$.

Definition 2.3.8. Let

$$\begin{array}{ccccc} M & \overset{e_\infty}{\rightarrowtail} & C_\infty & \overset{\iota_\infty}{\rightarrowtail} & C \\ & {\scriptstyle 1_M} \searrow & \downarrow {\scriptstyle s_\infty} & \swarrow {\scriptstyle s} & \\ & & M & & \end{array}$$

be the jet-factorisation of e in $\mathcal{E}/M$. Then the jet reflexive graph of $\mathbb{C}$ is the reflexive graph

$$\mathbb{C}_\infty = \left(C_\infty \;\underset{t \circ \iota_\infty}{\overset{s_\infty}{\rightrightarrows}}\!\!\!\!\!\!\!\!\xleftarrow{\;e_\infty\;}\; M \right)$$

in $\mathcal{E}$.

To equip this reflexive graph $\mathbb{C}_\infty$ with a composition operation we require a slight digression. To understand the reason for this digression we consider the special case that the base space $M = 1$. Then we can make the following straightforward argument. The arrow

$$C_\infty \times M \xrightarrow{1_{C_\infty} \times e_\infty} C_\infty \times C_\infty$$

is jet dense because (as an enriched factorisation system) the left class of the jet factorisation system is closed under tensor. Then we define the composition on C_∞ to be the unique lift of the following square

$$\begin{array}{ccc} C_\infty \times M & \xrightarrow{\pi_1} & C_\infty \\ {\scriptstyle 1_{C_\infty}\times e_\infty}\downarrow & \overset{\mu_\infty}{\nearrow} & \downarrow{\scriptstyle \iota_\infty} \\ C_\infty \times C_\infty & \xrightarrow{\mu\circ(\iota_\infty\times\iota_\infty)} & C \end{array}$$

and the associativity and unit axioms can be seen to hold. However if we now attempt to do the same thing in the slice category $\mathcal{E}/M$ we can still show that the arrow

$$(C_\infty, s_\infty) \xrightarrow{(1_{C_\infty}, e_\infty)} (C_\infty, t_\infty) \times (C_\infty, s_\infty) \cong (C_\infty \ {}_{t_\infty}\!\times_{s_\infty} C_\infty, t_\infty\pi_1)$$

is jet dense but there is no way to map out of $(C \ {}_t\!\times_s C, t\pi_1)$ using μ. The problem is that given arrows $f, g \in C$ such that $cod(f) = dom(g)$ the map $t\pi_1$ picks out the 'middle' object $cod(f)$ which cannot be specified from the composite $\mu(f, g)$ alone. We can rescue the idea of using a lift to define the composition by using the results of Section 2.2.1 to prove that the arrow

$$(C_\infty, s_\infty) \xrightarrow{(1_{C_\infty}, e_\infty t_\infty)} (C_\infty \ {}_{t_\infty}\!\times_{s_\infty} C_\infty, s_\infty\pi_1)$$

is jet dense in $\mathcal{E}/M$. Then we can proceed in an analogous fashion to the case $M = 1$.

The next Lemma tells us that the map which takes an arrow g of $\mathbb{C}_\infty$ and returns the composable pair $(g, 1_{cod(g)})$ in $2_\times C_\infty$ is jet dense over the source of g.

Lemma 2.3.9. *The arrow*

$$\begin{array}{ccc} C_\infty & \xrightarrow{(1_{C_\infty}, e_\infty t_\infty)} & 2_\times C_\infty \\ & {\scriptstyle s_\infty}\searrow \quad \swarrow{\scriptstyle s_\infty\circ\pi_1} & \\ & M & \end{array}$$

is jet-dense in $\mathcal{E}/M$.

Proof. The arrow

$$\begin{array}{ccc} M & \xrightarrow{e_\infty} & C_\infty \\ & {\scriptstyle 1_M}\searrow \quad \swarrow{\scriptstyle s_\infty} & \\ & M & \end{array}$$

in $\mathcal{E}/M$ is jet dense by the definition of jet part in Definition 2.3.8. Then by Corollary 2.2.9 the arrow

$$\begin{array}{ccc} C_\infty & \xrightarrow{(1_{C_\infty}, e_\infty t_\infty)} & 2_\times C_\infty \\ & {\scriptstyle 1_{C_\infty}} \searrow \quad \swarrow {\scriptstyle \pi_1} & \\ & C_\infty & \end{array}$$

obtained by pulling back along t_∞ is jet dense in $\mathcal{E}/C_\infty$. But now by Corollary 2.2.8 the arrow

$$\begin{array}{ccc} C_\infty & \xrightarrow{(1_{C_\infty}, e_\infty t_\infty)} & 2_\times C_\infty \\ & {\scriptstyle s_\infty} \searrow \quad \swarrow {\scriptstyle s_\infty \pi_1} & \\ & M & \end{array}$$

obtained by postcomposition by s_∞ is jet dense in $\mathcal{E}/M$ as required. □

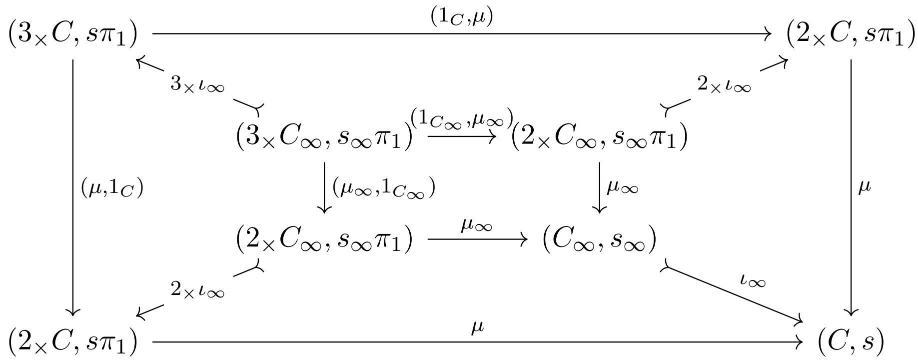
Now we are in a position to define a composition on the jet part of a category.

Corollary 2.3.10. *Let $\mathbb{C}$ be a category with composition $\mu : C\,{}_t{\times}_s\,C \to C$. Let $\mathbb{C}_\infty$ be the jet reflexive graph of of $\mathbb{C}$. Then we can make $\mathbb{C}_\infty$ into a category by defining the composition $\mu_\infty : C_\infty\,{}_t{\times}_s\,C_\infty \to C_\infty$ as the diagonal lift of the following diagram:*

$$\begin{array}{ccc} (C_\infty, s_\infty) & \xrightarrow{\quad 1_{C_\infty} \quad} & (C_\infty, s_\infty) \\ {\scriptstyle (1_{C_\infty}, e_\infty t_\infty)} \downarrow & \overset{\mu_\infty}{\nearrow} & \downarrow {\scriptstyle \iota_\infty} \\ (2_\times C_\infty, s_\infty \circ \pi_1) & \xrightarrow{\mu \circ (2_\times \iota_\infty)} & (C, s) \end{array}$$

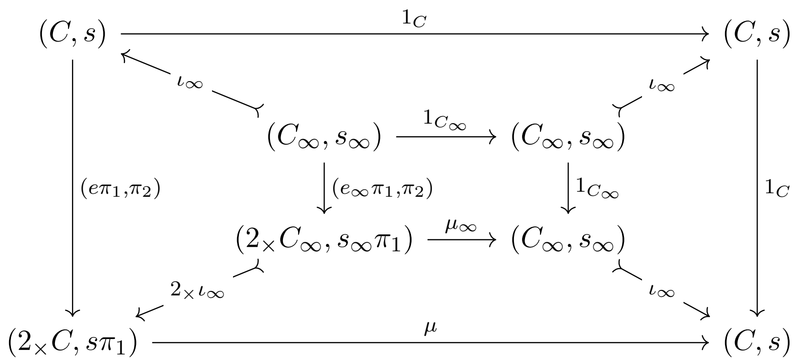
where ι_∞ is jet closed by the definition of C_∞ and $(1_{C_\infty}, e_\infty t_\infty)$ is jet dense by Lemma 2.3.9. We call the category $\mathbb{C}_\infty$ the jet part of $\mathbb{C}$.

Proof. The associativity of μ_∞ is inherited from the associativity of μ. To see this consider the diagram:



where the outer square commutes because μ is associative and the top, bottom, left and right squares commute using the definition of μ_∞ above. But this implies that the inner square commutes because ι_∞ is monic.

One of the unit laws for μ_∞ is already enforced by the upper commutative triangle in the definition of μ_∞. The other follows from combining the fact that ι_∞ is monic and that in the diagram:



the outer square commutes using a unit law for μ and the other squares are immediately seen to commute. $\square$

Example 2.3.11. Let $(R, +)$ be the group with arrow space R and composition given by addition. Then the jet part is $(\{(a \in R : 0 \approx a)\}, +)$. Using an argument similar to the one in Example 2.3.3 we see that actually $(R, +)_\infty \cong (D_\infty, +)$. The argument generalises without difficulty to show that the jet part of $(R^n, +)$ is $(D^n_\infty, +)$.

Example 2.3.12. Let (G, μ) be a Lie group whose underlying smooth manifold is n-dimensional. Since G is locally isomorphic to R^n we see that its jet part is a group of the form (D^n_∞, μ). Now to give a multiplication

$$\mu : D^n_\infty \times D^n_\infty \to D^n_\infty$$

is to give arrows

$$f_1, ..., f_n : (D_\infty)^{2n} \to R$$

taking values in nilpotent elements. Now we have that

$$(D_\infty)^{2n} = \bigcup_k (D_k)^{2n}$$

and so, since $\mathcal{E}(-, R)$ sends colimits to limits the hom-set $\mathcal{E}(D^{2n}_\infty, R)$ is given by the limit

$$... \to \mathcal{E}(D^{2n}_{k+1}, R) \to \mathcal{E}(D^{2n}_k, R) \to ...$$

which by the Kock-Lawvere axiom is equivalently the limit of the polynomial algebras

$$... \to \mathbb{R}[X_1, ..., X_{2n}]/I_{k+1} \to \mathbb{R}[X_1, ..., X_{2n}]/I_k \to ...$$

where I_k is the ideal generated by $(X^k_1, X^k_2, ..., X^k_{2n})$. This means that $\mathcal{E}(D^{2n}_\infty, R)$ can be identified with the ring $\mathbb{R}[[X_1, ..., X_{2n}]]$ of formal power series. Now the condition that the f_i take values in the nilpotent elements implies that the constant term of the power series p_i corresponding to f_i is zero. Under this correspondence, the group axioms for G correspond to the axioms making $p_1, ..., p_n$ into a formal group law. So the jet part of a classical Lie group encodes precisely its underlying formal group law, as described in the introduction.

Definition 2.3.13. Let $\mathbb{C}$ be a category in $\mathcal{E}$ and $\mathbb{C}_\infty$ be the category on its jet part as defined in Corollary 2.3.10. Then $\mathbb{C}$ is a jet category iff the inclusion $\mathbb{C}_\infty \rightarrowtail \mathbb{C}$ induced by ι_∞ is an isomorphism. We write $Cat_\infty(\mathcal{E})$ for the full subcategory of $Cat(\mathcal{E})$ on the jet categories. We abuse notation and write ι_∞ for both the inclusion $C_\infty \rightarrowtail C$ and the inclusion $\mathbb{C}_\infty \rightarrowtail \mathbb{C}$.

Lemma 2.3.14. *The function $(-)_\infty : Cat(\mathcal{E}) \to Cat_\infty(\mathcal{E})$ extends to a functor.*

Proof. Let $\phi : \mathbb{C} \to \mathbb{D}$ be a functor. Then the square

$$\begin{array}{ccc} (M, 1_M) & \xrightarrow{e^D_\infty \phi_0} & (D_\infty, s^D_\infty) \\ \Big\downarrow e^C_\infty & \overset{\phi_\infty}{\nearrow} & \Big\downarrow \iota^D_\infty \\ (C_\infty, s^C_\infty) & \xrightarrow{\phi_1 \iota^C_\infty} & (D, s^D) \end{array}$$

commutes in $\mathcal{E}$ and hence there exists a unique filler ϕ_∞. It is immediate from the definition that ϕ_∞ preserves identities. We now remark that in the cube

$$\begin{array}{ccccccc}
(2_\times C, s^C\pi_1) & & & \xrightarrow{\mu^C} & & & (C, s^C) \\
\downarrow{\scriptstyle 2_\times\phi_1} & \overset{2_\times\iota^C_\infty}{\nwarrow} & (2_\times C_\infty, s^C_\infty\pi_1) & \xrightarrow{\mu^C_\infty} & (C_\infty, s^C_\infty) & \overset{\iota^C_\infty}{\nearrow} & \downarrow{\scriptstyle \phi_1} \\
 & & \downarrow{\scriptstyle 2_\times\phi_\infty} & & \downarrow{\scriptstyle \phi_\infty} & & \\
 & \underset{2_\times\iota^D_\infty}{\swarrow} & (2_\times D_\infty, s^D_\infty\pi_1) & \xrightarrow{\mu^D_\infty} & (D_\infty, s^D) & \underset{\iota^D_\infty}{\searrow} & \\
(2_\times D, s^D\pi_1) & & & \xrightarrow{\mu^D} & & & (D, s^D)
\end{array}$$

the outer square commutes by functoriality of ϕ , the left and right faces commute by definition of ϕ_∞ and the top and bottom faces commute by the definition of μ_∞. Therefore the inner square commutes because the arrow ι^D_∞ is a monomorphism and hence ϕ_∞ preserves composition as required. $\square$

Proposition 2.3.15. *We have an adjunction $j \dashv (-)_\infty$ where j is the full inclusion $Cat_\infty(\mathcal{E}) \hookrightarrow Cat(\mathcal{E})$. In other words $Cat_\infty(\mathcal{E})$ is a coreflective subcategory of $Cat(\mathcal{E})$.*

Proof. Let $\mathbb{K}$ be a jet category; this means that the inclusion $\iota^K_\infty : \mathbb{K}_\infty \rightarrowtail \mathbb{K}$ is an isomorphism. We define the unit η by $\eta_\mathbb{K} = (\iota^K_\infty)^{-1}$. Let $\mathbb{C}$ be an arbitrary category in $\mathcal{E}$. We define the counit ε of the adjunction by $\varepsilon_\mathbb{C} = \iota^C_\infty$. Then $\varepsilon_{j(\mathbb{K})} \circ j(\eta_\mathbb{K}) = \iota^K_\infty \circ (\iota^K_\infty)^{-1} = 1_{j\mathbb{K}}$ and $(\varepsilon_\mathbb{C})_\infty \circ \eta_{\mathbb{C}_\infty} = (\iota^C_\infty)_\infty \circ (\iota^{C_\infty}_\infty)^{-1}$. But by definition of $(\iota^C_\infty)_\infty$ we see that

$$\begin{array}{ccccc}
M & \overset{e^{C_\infty}_\infty}{\rightarrowtail} & (\mathbb{C}_\infty)_\infty & \overset{\iota^{C_\infty}_\infty}{\rightarrowtail} & \mathbb{C}_\infty \\
\downarrow{\scriptstyle 1_M} & & \downarrow{\scriptstyle (\iota^C_\infty)_\infty} & & \downarrow{\scriptstyle \iota^C_\infty} \\
M & \underset{e^C_\infty}{\rightarrowtail} & \mathbb{C}_\infty & \underset{\iota^C_\infty}{\rightarrowtail} & \mathbb{C}
\end{array}$$

commutes and so $\iota^C_\infty \circ (\iota^C_\infty)_\infty \circ (\iota^{C_\infty}_\infty)^{-1} = \iota^C_\infty \circ (\iota^C_\infty)^{-1} \circ \iota^C_\infty = \iota^C_\infty$ and so $(\iota^C_\infty)_\infty \circ (\iota^{C_\infty}_\infty)^{-1} = 1_{\mathbb{C}_\infty}$ because ι^C_∞ is a monomorphism. $\square$

Remark 2.3.16. The category $Cat(\mathcal{E})$ of internal categories in $\mathcal{E}$ can be enriched over $\mathcal{E}$ by interpreting all the data in the internal logic of $\mathcal{E}$. Therefore for categories $\mathbb{C} = C \rightrightarrows M$ and $\mathbb{D} = D \rightrightarrows N$ the hom-object

$$Cat(\mathcal{E})(\mathbb{C}, \mathbb{D})$$

is the subobject of $D^C \times N^M$ in $\mathcal{E}$ defined by the proposition

$$\{(\psi, \phi) : (s\psi = \phi s) \wedge (t\psi = \phi t) \wedge (\psi e = e\phi) \wedge (\mu(2_\times \psi) = \psi\mu)\}$$

the identity arrow $id_{\mathbb{G}} : 1 \to Cat(\mathcal{E})(\mathbb{G}, \mathbb{G})$ is the global element $(1_G, 1_M)$ and the composition

$$Cat(\mathcal{E})(\mathbb{C}, \mathbb{D}) \times Cat(\mathcal{E})(\mathbb{D}, \mathbb{E}) \xrightarrow{\circ} Cat(\mathcal{E})(\mathbb{C}, \mathbb{E})$$

is defined by $(\psi_1, \phi_1) \circ (\psi_2, \phi_2) = (\psi_2\psi_1, \phi_2\phi_1)$. The axioms expressing the associativity and unit laws are proved in a completely analogous way to how the associativity and unit laws for ordinary functors are proved.

If $\mathbb{C}$ is a category in $\mathcal{E}$ and X an object of $\mathcal{E}$ then cotensor $\mathbb{C}^X$ is the category that has underlying reflexive graph

$$C^X \underset{t^X}{\overset{s^X}{\rightrightarrows}} M^X, \quad e^X : M^X \rightarrowtail C^X$$

and composition μ^X. Note that internal functors $\mathbb{D} \to \mathbb{C}^X$ are in bijection with internal functors $\dot{X} \times \mathbb{D} \to \mathbb{C}$ where $\dot{X}$ is the internal category with underlying reflexive graph

$$X \underset{1_X}{\overset{1_X}{\rightrightarrows}} X, \quad 1_X : X \rightarrowtail X$$

and the only possible composition. Henceforth we write $\mathbb{D}^{\mathbb{C}}$ for the object $Cat(\mathcal{E})(\mathbb{C}, \mathbb{D})$ seen as an object of $\mathcal{E}$.

Lemma 2.3.17. *Let $\mathbb{C}$ and $\mathbb{D}$ be internal categories in $\mathcal{E}$ and $\mathbb{D}_\infty$ and $\mathbb{C}_\infty$ be their jet parts. Then $\mathbb{C}^{\mathbb{D}_\infty} \cong \mathbb{C}_\infty^{\mathbb{D}_\infty}$ in $\mathcal{E}$.*

Proof. To show that $\mathbb{C}^{\mathbb{D}_\infty} \cong \mathbb{C}_\infty^{\mathbb{D}_\infty}$ it will suffice to show that for all representable objects X in $\mathcal{E}$ and internal functors $F : \mathbb{D}_\infty \times \dot{X} \to \mathbb{C}$ we have a unique lift G making

$$\begin{array}{ccc} & & \mathbb{C}_\infty \\ & \overset{G}{\nearrow} & \downarrow \iota_\infty^C \\ \mathbb{D}_\infty \times \dot{X} & \xrightarrow{F} & \mathbb{C} \end{array}$$

commute. But we can just take $G = F_\infty$ because the fact that $(-)_\infty$ is a right adjoint implies that $(\mathbb{D}_\infty \times \dot{X})_\infty = \mathbb{D}_\infty \times \dot{X}$. $\square$

In Section 2.2.2 we saw that the jet factorisation could be described using the relation $\approx$ in $\mathcal{E}/M$. This means that for preordered categories $\mathbb{C}$ in $\mathcal{E}$ the jet part $\mathbb{C}_\infty$ can also be described using the functor $\mathbb{N}$.

Lemma 2.3.18. *For any preorder $\mathbb{C}$ in $Cat(\mathcal{E})$ with object space M the jet part $\mathbb{C}_\infty$ is the preorder with arrow space $\mathbb{C}^{\mathbf{2}}_\infty = \mathbb{N}^{\mathbf{2}}_M \cap \mathbb{C}^{\mathbf{2}}$.*

Proof. Using Lemma 2.2.21 we have the following expression for the arrow space of $\mathbb{C}_\infty$:

$$(\mathbb{C}^{\mathbf{2}}_\infty, \pi_1) = \{(a,b) \in (C, \pi_1) :\ \exists m \in (M, 1_M).\ \Delta(m) \approx (a,b)\}$$

in the slice category $\mathcal{E}/M$. Now objects of $\mathcal{E}/M$ are arrows in $\mathcal{E}$ so an element $q \in M$ in $\mathcal{E}$ corresponds to stage of definition in $\mathcal{E}/M$. Let us fix a stage of definition $q \in M$. Then an element $(a,b) \in (C, \pi_1)$ in $\mathcal{E}/M$ corresponds to an element $(a,b) \in C$ in $\mathcal{E}$ such that $\pi_1(a,b) = q$. Similarly an element $m \in (M, 1_M)$ in $\mathcal{E}/M$ corresponds to an element $m \in M$ such that $m = q$. Therefore we have the equalities

$$\begin{aligned}
\mathbb{C}^{\mathbf{2}}_\infty &= \{(a,b,q) \in C \times M : (a = q) \wedge (\exists m \in M.\ (m = q) \wedge (\Delta(m) \approx (a,b)))\} \\
&= \{(a,b) \in C : a \approx b\} \\
&= \mathbb{N}^{\mathbf{2}}_M \cap \mathbb{C}^{\mathbf{2}}
\end{aligned}$$

as required. $\square$

In order to put the structure of a groupoid on the jet part $\mathbb{C}_\infty$ of a category $\mathbb{C}$ we require a little more than the assumption that $\mathbb{C}$ has the necessary additional structure and relations to make it a groupoid. For the rest of this section we fix a groupoid $\mathbb{G}$ that has underlying reflexive graph

$$G \underset{t}{\overset{s}{\rightrightarrows}} M \quad (e : M \to G)$$

and multiplication μ. We will identify a necessary condition for the jet part $\mathbb{G}_\infty$ of $\mathbb{G}$ to have groupoid structure; first we need a preparatory lemma.

Lemma 2.3.19. *Let $a, b \in_{(X,sa)} (G, s)$ such that $a \approx b$ at stage of definition (X, sa). Let $c \in_{(X,sc)} (G, s)$ such that $tc = sa(= sb)$. Then $ac \approx bc$ at stage of definition (X, sc).*

Proof. First we write down the data entailed by $a \approx b$ at stage of definition (X, sa). There is a cover $(\iota_i : (X_i, sa\iota_i) \to (X, sa))_{i \in I}$ and for all $i \in I$ the following holds. For notational convenience we write a_i for the restriction $a|_{(X_i,sa\iota_i)} = a\iota_i$. There exists a Weil spectrum D_i, an arrow

$$(X_i, sa_i) \times (M \times D_i, \pi_1) \xrightarrow{\phi_i} (G, s)$$

and an arrow $d_i : (X_i, sa_i) \to (M \times D_i, \pi_1)$ such that $\phi(1_{X_i}, 0) = a_i$ and $\phi(1_{X_i}, d_i) = b_i$. Then the data consisting of the cover $(\iota_i)_{i \in I}$ and for each i the Weil spectrum D_i, the arrow

$$\phi_i' = (X_i \ {}_{sc_i}\!\times_{\pi_1} (M \times D_i), sc_i\pi_1) \xrightarrow{(c_i\pi_1, \phi_i(\pi_1,(sa_i,\pi_2)))} (2_\times G, s\pi_1)$$

and the arrow

$$d_i' = (X_i, sc_i) \xrightarrow{(sc_i, \pi_2 d_i)} (M \times D_i, \pi_1)$$

tells us that $(c, a) \approx (c, b)$. Indeed we have the equalities

$$\phi_i(\pi_1, (sa_i, \pi_2\pi_2)) \circ (1_{X_i}, (sc_i, \pi_2 d_i)) = \phi(1_{X_i}, (sa_i, \pi_2 d_i)) = \phi(1_{X_i}, d_i) = b_i$$

$$\phi_i(\pi_1, (sa_i, \pi_2\pi_2)) \circ (1_{X_i}, (sc_i, \pi_2 0)) = \phi(1_{X_i}, (sa_i, \pi_2 0)) = \phi(1_{X_i}, 0) = a_i$$

and

$$c\iota_i\pi_1 \circ (1_{X_i}, (sc\iota_i, \pi_2 d_i)) = c\iota_i 1_{X_i} = c_i$$

whence $(c_i, a_i) \approx (c_i, b_i)$. But now the result follows from Lemma 2.2.15 by applying μ. □

Proposition 2.3.20. *Let $\mathbb{G}$ be a groupoid in $\mathcal{E}$ with arrow space G and object space M. Suppose further that $\mathbb{G}_\infty$ is a groupoid. Then the relation $\approx$ is symmetric on (G, s) in $\mathcal{E}/M$.*

Proof. Let $a, b \in_{(X,sa)} (G, s)$ such that $a \approx b$ at stage of definition (X, sa). Then $a^{-1} \in_{(X,ta)} (G, s)$ has $ta^{-1} = sa(= sb)$. So by Lemma 2.3.19 we have that $eta = aa^{-1} \approx ba^{-1}$ at stage of definition (X, ta) and hence $ba^{-1} \in_{(X,ta)}$

(G_∞, s_∞). Since $\mathbb{G}_\infty$ is a groupoid we have that $ab^{-1} \in_{(X,ta)} (G_\infty, t_\infty)$ also and hence $ab^{-1} \in_{(X,tb)} (G_\infty, s_\infty)$.

This means that $etb \approx ab^{-1}$ at stage of definition (X, tb). Now we note that $b \in_{(X,sb)} (G, s)$ has $tb = setb = sab^{-1}$ and so by Lemma 2.3.19 again we deduce that $b \approx ab^{-1}b = a$ as required. □

Now we give a counterexample which shows that we cannot immediately specialise the jet part construction for categories to construct a jet part for an arbitrary groupoid in $\mathcal{E}$. We will use one of the simplest non-classical groupoids we have at our disposal: the pair groupoid ∇D where $D = \{x \in R : x^2 = 0\}$. Using Lemma 2.3.18 we see that

$$(\nabla D)^{\mathbf{2}}_\infty = \{(a, b) \in D \times D : a \approx b\}$$

in $\mathcal{E}$ and so to show that $(\nabla D)^{\mathbf{2}}_\infty$ is not a groupoid it will suffice to show that $\approx$ is not symmetric on D in $\mathcal{E}$. To prove this we will show that any jet starting from the generalised element 1_D must be trivial. The intuitive reason for this is that D is not closed under addition and so there is no more 'space' for the jet to move into.

Lemma 2.3.21. *The relation $\approx$ is not symmetric on D.*

Proof. Let us consider the generalised elements at stage D described by $0 : D \to D$ and 1_D. It will suffice to show that $0 \sim 1_D$ but not $1_D \approx 0$. To see that $0 \sim 1_D$ we choose $D_W = D$, $\phi = 1_D$ and $d = 1_D$. Then $\phi(0) = 0$ and $\phi(d) = 1_D$.

To show that $1_D \approx 0$ does not hold it will suffice to show that for all elements f such that $1_D \sim f$ then necessarily $f = 1_D$. So let us suppose that we have an f such that $1_D \sim f$. Since the only covers of D are trivial this would mean that there exist $D_W \in Spec(Weil)$, $\phi : D \times D_W \to D$ and $d : D \to D_W$ such that $\phi(x, 0) = x$ and $\phi(x, d(x)) = f(x)$ for all $x \in D$. Let w be the number of indeterminates in the polynomial defining the Weil presentation W. Now in a similar manner to Lemma 1.1.3 we use Hadamard's Lemma twice and the fact that D is defined by the formula $x^2 = 0$ to see that

$$\phi(x_1, \vec{x}) \cong \phi_0(\vec{x}) + x_1\phi_1(\vec{x})$$

for some smooth functions $\phi_0, \phi_1 : \mathbb{R}^w \to \mathbb{R}$. Now the equation $\phi(a, 0) = a$ tells us that

$$\phi_0(0) + x_1\phi_1(0) = \phi(x_1, 0) = x_1$$

and so $\phi_0(0) = 0$ and $\phi_1(0) = 1$. Hence by Hadamard's Lemma we see that

$$\phi_1(\vec{x}) = 1 + \Sigma_{i=2}^{w+1} x_i\psi_i(\vec{x})$$

for some $\psi_i : \mathbb{R}^w \to \mathbb{R}$. But since for all i there is an equality of the form $x_i^{k_i} = 0$ in W we see that $N = \Sigma_{i=2}^{w+1} x_i\psi_i(\vec{x})$ is nilpotent of degree $n = \Sigma_{i=2}^{w+1} k_i$. (This follows from the pigeonhole principle.) Therefore the arrow

$$i_\phi = \Sigma_{j=0}^{n-1} (-1)^j N^k : D_W \to D$$

is a pointwise multiplicative inverse for ϕ_1. Now because ϕ has codomain D we must have that

$$\phi_0(\vec{x})^2 + 2x_1\phi_0(\vec{x})\phi_1(\vec{x}) = \phi(x_1, \vec{x})^2 = 0$$

and so $\phi_0(\vec{x})\phi_1(\vec{x}) = 0$. But since ϕ_1 has a pointwise multiplicative inverse this means that $\phi_0(\vec{x}) = 0$ and so $\phi(x_1, \vec{x}) \cong x_1\phi_1(\vec{x})$. Similarly we see that

$$d(x) \cong \vec{a} + \vec{b}x$$

where $(a_i + b_i x)^{k_i} = 0$ when $x^2 = 0$. But since $a_i \in \mathbb{R}$ we see that $a_i = 0$ and hence

$$\phi(x, d(x)) = x + x\Sigma_{i=2}^{w+1} d(x)_i\psi_i(d(x)) = x + x\Sigma_{i=2}^{w+1} b_i x\psi_i(d(x)) = x$$

and we deduce that $f = 1_D$ as required. □

Corollary 2.3.22. *The jet part $(\nabla D)_\infty$ of the pair groupoid ∇D is not a groupoid.*

Proof. The result immediately follows from Lemma 2.3.21 and the remarks preceding it. □

Fortunately the condition that the relation $\approx$ is symmetric on (G, s) in $\mathcal{E}/M$ is not only necessary but also sufficient to ensure that the jet part $\mathbb{G}_\infty$ of $\mathbb{G}$ is a groupoid.

Lemma 2.3.23. *Let $a \in (G,s)$ such that $esa \approx a$ in (G,s). Suppose further that $\approx$ is symmetric on (G,s). Then $eta \approx a^{-1}$ in (G,s).*

Proof. Since $\approx$ is symmetric we have that $a \approx esa$ and $ta^{-1} = sa$. So by Lemma 2.3.19 we have that $eta \approx a^{-1}$. □

Lemma 2.3.24. *Let $a, b \in (G,s)$ such that $a \approx b$ in (G,s). Then $a^{-1} \approx b^{-1}$ in (G,t).*

Proof. Immediate from Lemma 2.2.15. □

Corollary 2.3.25. *If $\approx$ is symmetric on (G,s) then the arrow*

$$e_\infty : (M, 1_M) \to (G_\infty, t_\infty)$$

is jet dense.

Proof. Let $a \in (G_\infty, t_\infty)$; by definition of G_∞ this means that that $esa \approx a$ in (G,s). Since $\approx$ is symmetric on (G,s) we have that $eta \approx a^{-1}$ and then $eta \approx a$ in (G,t) from Lemmas 2.3.23 and 2.3.23 respectively. Hence the arrow $e_\infty : (M, 1_M) \to (G_\infty, t_\infty)$ is jet dense as required. □

Proposition 2.3.26. *Let $\mathbb{G}$ be a groupoid in $\mathcal{E}$ such that the relation $\approx$ is symmetric on the object (G_∞, s_∞) in $\mathcal{E}/M$. Then the jet part $\mathbb{G}_\infty$ can be given the structure of a groupoid.*

Proof. By Corollary 2.3.25 we see that the left arrow in the square

$$\begin{array}{ccc}
(M, 1_M) & \xrightarrow{e_\infty} & (G_\infty, s_\infty) \\
\Big\downarrow{\scriptstyle e_\infty} & \overset{i_{G_\infty}}{\nearrow} & \Big\downarrow{\scriptstyle \iota_\infty} \\
(G_\infty, t_\infty) & \xrightarrow{i_G \iota_\infty} & (G,s)
\end{array}$$

is jet dense. This means that there is an unique filler i_{G_∞} which we will take as the inverse for the jet part $\mathbb{G}_\infty$. Since the equations $s_\infty i_{G_\infty} = t_\infty$ and $t_\infty i_{G_\infty} = s_\infty$ are immediately seen to hold it remains to check that the inverse axioms hold. So observe that in the diagram:

$$\begin{array}{ccccc}
(G_\infty, s_\infty) & \xrightarrow{(1_{G_\infty}, i_{G_\infty})} & (2_* G_\infty, s_\infty \pi_1) & \xrightarrow{\mu_\infty} & (G_\infty, s_\infty) \\
\Big\downarrow{\scriptstyle \iota_\infty} & & \Big\downarrow{\scriptstyle 2_* \iota_\infty} & & \Big\downarrow{\scriptstyle \iota_\infty} \\
(G,s) & \xrightarrow{(1_G, i_G)} & (2_* G, s\pi_1) & \xrightarrow{\mu} & (G,s)
\end{array}$$

the right-hand square commutes by the definition of μ_∞ in Definition 2.3.10 and the left-hand square commutes by the definition of i_{G_∞} above. But now we notice that the bottom row is equal to 1_G because i_G is an inverse for the multiplication μ; hence the top row is equal to 1_{G_∞} because ι_∞ is monic. Similarly the diagram

$$\begin{array}{ccccc} (G_\infty, s_\infty) & \xrightarrow{(i_{G_\infty}, 1_{G_\infty})} & (2_* G_\infty, s_\infty \pi_1) & \xrightarrow{\mu_\infty} & (G_\infty, s_\infty) \\ \downarrow{\scriptstyle \iota_\infty} & & \downarrow{\scriptstyle 2_*\iota_\infty} & & \downarrow{\scriptstyle \iota_\infty} \\ (G, s) & \xrightarrow{(i_G, 1_G)} & (2_* G, s\pi_1) & \xrightarrow{\mu} & (G, s) \end{array}$$

shows that the other inverse axiom holds. □

2.3.3 The Integral Factorisation System

We will define the integral factorisation system for categories in $\mathcal{E}$ but to motivate its introduction let us first consider a little of the classical theory of Lie groupoids. In Lie theory we have a situation where the arrows infinitesimally close to the identity arrows completely determine all the rest of the arrows. For this to be possible we must have a method of constructing macroscopic data from infinitesimal data. As a first attempt we will use the arrows infinitesimally close to the identity arrows to determine a vector field and appeal to a theorem asserting the local existence of solutions to vector fields.

Let us see how this would work out in the classical case. Fix a Lie groupoid $\mathbb{G}$ in the category Man with base space M and structure maps s, t and e. A time-dependent left-invariant vector field on $\mathbb{G}$ is one which is completely determined by a collection of vectors $(v_x(a))_{x \in M}$ depending smoothly on a time parameter $a \in I$ where each $v_x(a)$ is based at ex and tangent to the submanifold $s^{-1}x$. The value of the vector field at a non-identity element $f : x \to y$ is given by

$$(dL_f)v_y(a)$$

where $L_f : s^{-1}(y) \to s^{-1}(x)$ is precomposition and d denotes the derivative. In classical differential geometry a solution to this vector field is simply a solution $\gamma : I \to G$ to the differential equation

$$\gamma'(a) = (dL_{\gamma(a)})v_{t\gamma(a)}(a)$$

Let us now think about integrating such a vector field in terms of synthetic differential geometry. For each $x \in M$ we think of $v_x(a)$ as an infinitesimally small arrow with source x. For our solution we would like a source constant arrow $\psi_1 : I \to G$ satisfying

$$\psi_1(a+d) = v_{t\psi_1(a)}(a) \circ \psi_1(a) \tag{2.5}$$

for all $a \in I$ and all $d \in D$ and $\psi(0) = 1_x$. Thus at time a we can see that the target of the arrow $\psi(a)$ that we start with influences the arrow $v_{t\psi(a)}(a)$ that we want to postcompose with.

This observation suggests that the integration may be broken down into two steps. The first step integrates the vector field $dt(v_y(a))$ on the base space M to a path $\psi_0 : I \to M$. At this point we can throw away all of the vectors $v_y(a)$ apart from those of the form $v_{\psi_0(a)}(a)$. The second step integrates these $v_{\psi_0(a)}(a)$ to obtain a path $\psi_1 : I \to G$. However at this point we notice that the result of the first step can in fact be expressed in terms of infinitesimal data if we use jet groupoids (in the classical case Lie algebroids are used) rather than vector fields. This idea is made concrete in the following definition.

Definition 2.3.27. An A-path in a groupoid $\mathbb{G}$ in $\mathcal{E}$ is a groupoid homomorphism $\phi : (\nabla I)_\infty \to \mathbb{G}$.

Remark 2.3.28. The notion of A-path is a well-known one in the theory of Lie groupoids and we will see in Chapter 5 that our definition coincides with the classical one.

A groupoid homomorphism $\phi : (\nabla I)_\infty \to \mathbb{G}$ corresponding to a vector field $v_y(a)$ would have underlying path $\phi_0 = \psi_0$ and $\phi(a \to a+d)$ would pick out the infinitesimal arrow corresponding to the tangent vector $v_{\psi_0(a)}(a)$. Therefore finding a solution ψ now amounts to solving

$$\psi(a+d) = \phi(a \to a+d) \circ \psi(a)$$

and hence we assert that the integral of the A-path ϕ should be a groupoid homomorphism $\psi : \nabla I \to \mathbb{G}$ such that the diagram

$$\begin{array}{ccc} (\nabla I)_\infty & \xrightarrow{\ \phi\ } & \mathbb{G} \\ {\scriptstyle \iota_\infty}\downarrow & \nearrow_{\psi} & \\ \nabla I & & \end{array}$$

commutes.

In fact we can work with categories rather than groupoids by replacing the arrow $\iota_\infty : \nabla I_\infty \to \nabla I$ with the arrow $\iota_\infty : \mathbb{I}_\infty \to \mathbb{I}$. We define the integral factorisation system by using the arrow ι_∞ to generate the left class and then define a natural completion operation. A standard categorical argument (see Section 2 of [6]) shows that the subcategory defined as the image of this completion operation is a reflective subcategory of $Cat(\mathcal{E})$. It consists of all the categories $\mathbb{C}$ for which every A-path admits a unique integral.

Definition 2.3.29. Let Σ be the singleton class of arrows in $Cat(\mathcal{E})$ given by

$$\Sigma := \{\iota_\infty : \mathbb{I}_\infty \xrightarrow{\iota_\infty} \mathbb{I}\}$$

then the integral factorisation system is the $Cat(\mathcal{E})$-factorisation system

$$(L_{int}, R_{int}) = (Sat(\tilde{\Sigma} \cup \delta^o(\tilde{\Sigma})), \Sigma^{\perp_\mathcal{V}})$$

on $Cat(\mathcal{E})$ that is generated using Corollary 2.1.40. Note that $Cat(\mathcal{E})$ is locally presentable so the conditions of Corollary 2.1.40 are satisfied.

To put this in more concrete terms: an arrow $r : \mathbb{X} \to \mathbb{Y}$ is in the right class R_{int} (and is called integral closed) iff the following square is a pullback of categories:

$$\begin{array}{ccc} \mathbb{X}^{\mathbb{I}} & \xrightarrow{\mathbb{X}^{\iota_\infty}} & \mathbb{X}^{\mathbb{I}_\infty} \\ \downarrow{\scriptstyle r^{\mathbb{I}}} & & \downarrow{\scriptstyle r^{\mathbb{L}_\infty}} \\ \mathbb{Y}^{\mathbb{I}} & \xrightarrow{\mathbb{Y}^{\iota_\infty}} & \mathbb{Y}^{\mathbb{I}_\infty} \end{array}$$

and an arrow $l : \mathbb{A} \to \mathbb{B}$ is in the left class L_{int} iff for all $r \in R_{int}$ the following square is a pullback:

$$\begin{array}{ccc} \mathbb{X}^{\mathbb{B}} & \xrightarrow{\mathbb{X}^{l}} & \mathbb{X}^{\mathbb{A}} \\ \downarrow{\scriptstyle r^{\mathbb{B}}} & & \downarrow{\scriptstyle r^{\mathbb{A}}} \\ \mathbb{Y}^{\mathbb{B}} & \xrightarrow{\mathbb{Y}^{l}} & \mathbb{Y}^{\mathbb{A}} \end{array}$$

Definition 2.3.30. The integral completion $\mathbb{C}_{int}$ of a category $\mathbb{C}$ is the meditating object of the integral factorisation of the unique arrow $! : \mathbb{C} \to 1$:

$$\begin{array}{ccccc} \mathbb{C} & & \xrightarrow{\;\;!\;\;} & & 1 \\ & \searrow^{\tau} & & \nearrow_{!} & \\ & & \mathbb{C}_{int} & & \end{array}$$

An integral complete category is a category $\mathbb{C}$ for which $\tau : \mathbb{C} \to \mathbb{C}_{int}$ is an isomorphism and we write $Cat_{int}(\mathcal{E})$ for the full subcategory on integral complete categories.

Lemma 2.3.31. *The function $(-)_{int} : Cat(\mathcal{E}) \to Cat_{int}(\mathcal{E})$ extends to a functor.*

Proof. This is immediate by functoriality of factorisation. □

Proposition 2.3.32. *We have an adjunction $(-)_{int} \dashv k$ where k is the full inclusion $Cat_{int}(\mathcal{E}) \hookrightarrow Cat(\mathcal{E})$. In other words $Cat_{int}(\mathcal{E})$ is a reflective subcategory of $Cat(\mathcal{E})$.*

Proof. Let $\mathbb{C}$ be an arbitrary category in $\mathcal{E}$. Then we define the unit η by $\eta_{\mathbb{C}} = \tau^C : \mathbb{C} \to \mathbb{C}_{int}$. Let $\mathbb{X}$ be a integral complete category; this means that the arrow $\tau^X : \mathbb{X} \to \mathbb{X}_{int}$ is an isomorphism. So we define the counit ε by $\varepsilon_{\mathbb{X}} = (\tau^X)^{-1}$. Then $\varepsilon_{(\mathbb{C}_{int})} \circ (\eta_{\mathbb{C}})_{int} = (\tau^{C_{int}})^{-1} \circ (\tau^C)_{int}$ and $k(\varepsilon_{\mathbb{X}}) \circ \eta_{k(\mathbb{X})} = k((\tau^X)^{-1}) \circ \tau^{k(\mathbb{X})} = 1_{k(\mathbb{X})}$. But we defined $(\tau^C)_{int}$ as the unique filler of the Diagram

$$\begin{array}{ccccc} \mathbb{C} & \xrightarrow{\tau^C} & \mathbb{C}_{int} & \xrightarrow{!} & 1 \\ \downarrow{\scriptstyle \tau^C} & & \downarrow{\scriptstyle (\tau^C)_{int}} & & \downarrow{\scriptstyle 1_1} \\ \mathbb{C}_{int} & \xrightarrow{\tau^{C_{int}}} & (\mathbb{C}_{int})_{int} & \xrightarrow{!} & 1 \end{array}$$

but since $\tau^{C_{int}}$ can fill this diagram also we see that $(\tau^{C_{int}})^{-1} \circ (\tau^C)_{int} = 1_{\mathbb{C}_{int}}$ as required. □

Remark 2.3.33. It is routine to modify the definitions of integral factorisation system on $Cat(\mathcal{E})$ and integral complete category to obtain an integral factorisation system on $Gpd(\mathcal{E})$ and a definition of integral complete groupoid. We now give a summary of the results.

Definition 2.3.34. Let Σ be the singleton class of arrows in $Gpd(\mathcal{E})$ given by

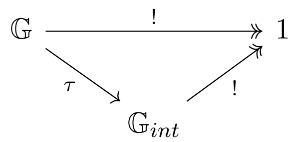
$$\Sigma := \{\iota_\infty : (\nabla I)_\infty \xrightarrow{\iota_\infty} (\nabla I)\}$$

then the integral factorisation system is the $Gpd(\mathcal{E})$-factorisation system

$$(L_{int}, R_{int}) = (Sat(\tilde{\Sigma} \cup \delta^o(\tilde{\Sigma})), \Sigma^{\perp_{\mathcal{V}}})$$

on $Gpd(\mathcal{E})$ that is generated using Corollary 2.1.40.

Definition 2.3.35. The integral completion $\mathbb{G}_{int}$ of a groupoid $\mathbb{G}$ is the meditating object of the integral factorisation of the unique arrow $! : \mathbb{G} \to 1$:



An integral complete groupoid is a category $\mathbb{G}$ for which $\tau : \mathbb{G} \to \mathbb{G}_{int}$ is an isomorphism and we write $Gpd_{int}(\mathcal{E})$ for the full subcategory on integral complete groupoids.

Proposition 2.3.36. *The full subcategory $Gpd_{int}(\mathcal{E}) \subset Gpd(\mathcal{E})$ on integral complete groupoids is a reflective subcategory.*

Chapter 3

Paths in Categories

The method that will be used in the proof of the synthetic version of Lie's second theorem is an axiomatisation and generalisation of an established proof in classical Lie theory. The central idea of this proof is to establish a relationship between paths in the Lie group G starting at the identity element e and paths in the Lie algebra $\mathfrak{g}$. We will always assume that our paths are smooth maps. To obtain a path $\delta \in \mathfrak{g}^I$ from a path $\gamma \in G^I$ with $\gamma(0) = e$ we take the tangent vectors $\gamma'(a)$ for $a \in I$ and use the derivative of left multiplication by $\gamma(a)^{-1}$ to map them to the origin

$$\delta(a) = (DL_{\gamma(a)^{-1}})_{\gamma(a)}\gamma'(a)$$

In the other direction if we are given $\delta \in \mathfrak{g}^I$ we obtain an element $\gamma \in G^I$ by solving the differential equation

$$\gamma'(a) = (DL_{\gamma(a)})_e\delta(a)$$

where we have that $\gamma(0) = e$ by construction. Furthermore in addition to being able to describe the elements of G^I in terms of elements of $\mathfrak{g}^I$ we can also describe the homotopies between elements of G^I using only infinitesimal linear data contained in the Lie algebra $\mathfrak{g}$. The proof is finished by observing that homotopy classes of paths in a simply connected Lie group G starting at e are in bijection with elements of G. An advantage of this proof is that it makes clear where the condition that the Lie group is simply connected is used. The full details can be found in [22] and an accessible explanation is in Chapter 5 of the Lie Groups section of [5].

We now describe an established generalisation of the above ideas to the theory of Lie groupoids. Given a Lie groupoid $\mathbb{G}$ with source map $s : G \to M$ we define the set of $\mathbb{G}$-paths to be

$$Path(\mathbb{G}) = \{\gamma \in G^I : \forall a \in I.\ (s\gamma(a) = s\gamma(0)) \wedge (\gamma(0) = es\gamma(0))\}$$

and given a Lie algebroid $\pi : E \to M$ with anchor ρ we define the set of E-paths to be

$$Path(E) = \{\delta \in E^I : \forall a \in I.\ \rho\delta(a) = \frac{d}{da}\pi\delta(a)\}$$

Then we can establish a bijection between $Path(\mathbb{G})$ and $Path(\mathfrak{g})$ in the case that $\mathfrak{g}$ is the Lie algebroid of $\mathbb{G}$. Similarly one can show that the natural homotopy relation on $Path(\mathbb{G})$ can be described in terms of a relation R on $Path(\mathfrak{g})$. Therefore we can form a groupoid $W(\mathfrak{g})$ that has arrow space defined by the quotient

$$Path(\mathfrak{g})/R$$

The details can be found in the paper [7]. The bijection between $Path(\mathbb{G})$ and $Path(\mathfrak{g})$ is Proposition 1.1. The groupoid $W(\mathfrak{g})$ is called the Weinstein groupoid and is its construction given in Section 2.1.

In this chapter we will transfer this theory to the setting where we compare integral complete categories in some well-adapted model of synthetic differential geometry $\mathcal{E}$ with jet categories in $\mathcal{E}$. This requires us to identify appropriate notions of connectedness for categories in $\mathcal{E}$ and describe a smooth concatenation operation. The notion corresponding to a $\mathbb{G}$-path is simply an internal functor $\mathbb{I} \to \mathbb{C}$ and the theory proceeds in an analogous fashion to the classical case. We obtain a path category analogous to the path groupoid of elementary topology in Section 3.4. The constructions corresponding to the Weinstein groupoid however permit some simplification. We exploit the existence of infinitesimal arrows to define the notion corresponding to a $\mathfrak{g}$ path as an internal functor

$$\mathbb{I}_\infty \to \mathbb{C}$$

where $\mathbb{I}_\infty$ is the jet part of $\mathbb{I}$ as defined in Section 2.3.2. This means that in Section 3.5 we can define the Weinstein category of an arbitrary category $\mathbb{C}$ in $\mathcal{E}$.

In the proof of Lie's second theorem that we give in Chapter 4 it will be convenient to work with the various notions of path directly rather than as equivalence classes of paths. To do this will require us to define a certain type of algebraic theory that we will call the theory of precategories.

3.1 Precategories and their Quotient Categories

Recall that for a topological space X we may form its path space X^I and define a concatenation operation $\mu : X^{I\,{}_1{+}_0\, I} \to X^I$ using a continuous reparametrisation map $I \to I\ {}_1{+}_0\ I$. In general this will not give the space X^I the structure of a groupoid because it will not in general be possible to choose μ to be associative. All the different n-fold composites are however homotopic and so if we take the quotient of X^I by identifying all paths that are homotopic (with fixed endpoints) then we do obtain a groupoid.

The corresponding problem must be solved in the analogous situation in which $\mathbb{I}$ replaces I and $Cat(\mathcal{E})$ replaces Top. However in the sequel it will be convenient to delay taking the quotient described above in order to compute a certain pullback in terms of paths rather than equivalence classes of paths. Therefore in Section 3.3 we define an operation $\mu : \mathbb{C}^{2*\mathbb{I}} \to \mathbb{C}$ that exhibits $\mathbb{C}^{\mathbb{I}}$ as a model of a certain algebraic theory that we will call the theory of precategories. Intuitively a precategory is a reflexive graph

$$C \underset{t}{\overset{s}{\underset{\longleftarrow e\, -}{\overrightarrow{\quad}}}}\!\!\!\!\!\!\longrightarrow M$$

that has an operation $\mu : C\ {}_t{\times}_s\ C \to C$ which is only a composition operation up to certain equivalence relations. To decide which arrows get identified the definition of precategory includes in addition the data of another reflexive graph

$$R \underset{r_2}{\overset{r_1}{\underset{\longleftarrow c\, -}{\overrightarrow{\quad}}}}\!\!\!\!\!\!\longrightarrow C$$

and we assert that if $f, g \in C$ such that $sf = sg$ and $tf = tg$ then f is equivalent to g iff there exists an $r \in R$ such that $r_1 r = f$ and $r_2 r = g$.

In this section we give the abstract definition of a precategory and a general quotienting procedure to obtain a category from a precategory. At the end we give the appropriate commutative diagrams that show that in the situation in which ∇I, rather than $\mathbb{I}$, replaces I and $Gpd(\mathcal{E})$, rather than $Cat(\mathcal{E})$, replaces Top we can define pregroupoids and their quotients in a similar manner.

Notation 3.1.1. Recall that we write $n_\times C = C \; {}_t\!\times_s \cdots {}_t\!\times_s C$. For an arrow $f : C \to D$ then we write $n_\times f$ for the induced arrow $n_\times C \to n_\times D$.

Definition 3.1.2. The category $\mathcal{G}$ is the finitely complete category freely generated by the arrows

$$R \;\begin{array}{c} \xrightarrow{\;r_1\;} \\ \xleftarrow{\;c\;} \\ \xrightarrow[\;r_2\;]{} \end{array}\; C \;\begin{array}{c} \xrightarrow{\;s\;} \\ \xleftarrow{\;e\;} \\ \xrightarrow[\;t\;]{} \end{array}\; M$$

satisfying the usual relations for a truncated globular object: $se = 1_M = te$, $r_1c = 1_G = r_2c$, $sr_1 = sr_2$ and $tr_1 = tr_2$.

Definition 3.1.3. The category Pre is the finitely complete category freely generated from $\mathcal{G}$ by adding arrows m, n, a, λ and ρ that satisfy the equations $sm = s\pi_1$ and $tm = t\pi_2$ (where π_1 and π_2 are the pullback projections $2_\times C \to C$) and that make the diagrams

$$\begin{array}{ccc} R \;{}_{r_1t}\!\times_{r_2s} R & \underset{(r_2\pi_1, r_2\pi_2)}{\overset{(r_1\pi_1, r_1\pi_2)}{\rightrightarrows}} & 2_\times C \\ \downarrow{\scriptstyle n} & & \downarrow{\scriptstyle m} \\ R & \underset{r_2}{\overset{r_1}{\rightrightarrows}} & C \end{array}, \qquad \begin{array}{ccc} & \overset{a}{\swarrow\!\!-\,-} & 3_\times C \\ & & {\scriptstyle m(m\times 1)}\downarrow\downarrow{\scriptstyle m(1\times m)} \\ R & \underset{r_2}{\overset{r_1}{\rightrightarrows}} & C \end{array},$$

$$\begin{array}{ccc} & \overset{\lambda}{\swarrow\!\!-\,-} & C \\ & & {\scriptstyle 1_C}\downarrow\downarrow{\scriptstyle m(1\times e)} \\ R & \underset{r_2}{\overset{r_1}{\rightrightarrows}} & C \end{array} \quad \textit{and} \quad \begin{array}{ccc} & \overset{\rho}{\swarrow\!\!-\,-} & C \\ & & {\scriptstyle 1_C}\downarrow\downarrow{\scriptstyle m(e\times 1)} \\ R & \underset{r_2}{\overset{r_1}{\rightrightarrows}} & C \end{array}$$

serially commutative.

Definition 3.1.4. A precategory in $\mathcal{E}$ is a finite limit preserving functor $Pre \to \mathcal{E}$. The category of precategories in $\mathcal{E}$ is the category $[Pre, \mathcal{E}]_{lex}$ and will sometimes be written $PreCat(\mathcal{E})$. Here the subscript lex indicates that we consider only functors that preserve finite limits.

Now we describe how to form a category from a precategory by identifying all the arrows $f, g \in C$ for which there exists $r \in R$ such that $r_1r = f$ and $r_2r = g$.

Definition 3.1.5. Let $F : Pre \to \mathcal{E}$ be a precategory in $\mathcal{E}$. Then the quotient reflexive graph $\overline{F}$ of F is the reflexive graph in $\mathcal{E}$ that has object space $F(M)$, arrow space $\bar{F}$ defined by the coequaliser

$$FR \underset{Fr_2}{\overset{Fr_1}{\rightrightarrows}} FC \overset{q}{\twoheadrightarrow} \overline{F}$$

and reflexive graph structure $\overline{s}$, $\overline{t}$ and $\overline{e}$ given by the factorisations of Fs, Ft and Fe through the coequaliser defining $\overline{F}$

$$\begin{array}{ccccc} FR & \underset{Fr_2}{\overset{Fr_1}{\rightrightarrows}} & FC & \overset{q}{\twoheadrightarrow} & \overline{F} \\ {\scriptstyle Fsr_1}\downarrow \; \uparrow{\scriptstyle Fce} \; \downarrow{\scriptstyle Ftr_1} & & {\scriptstyle Fs}\downarrow \; \uparrow{\scriptstyle Fe} \; \downarrow{\scriptstyle Ft} & & {\scriptstyle \overline{s}}\downarrow \; \uparrow{\scriptstyle \overline{e}} \; \downarrow{\scriptstyle \overline{t}} \\ FM & \underset{1_{FM}}{\overset{1_{FM}}{\rightrightarrows}} & M & \overset{1_{FM}}{\longrightarrow} & FM \end{array}$$

Lemma 3.1.6. *Let F be a precategory in $\mathcal{E}$. Let $n_\times C = C\ {}_s\!\times_t C\ {}_s\!\times_t \cdots {}_s\!\times_t C$ where there are n copies on the right hand side. Similarly for $n_\times \bar{F}$, $n_\times R$ and $n_\times q$. Then*

$$n_\times FR \underset{n_\times Fr_2}{\overset{n_\times Fr_1}{\rightrightarrows}} n_\times FC \overset{n_\times q}{\twoheadrightarrow} n_\times \bar{F}$$

is a coequaliser in $\mathcal{E}$.

Proof. First observe that

$$FR \; \begin{array}{c} \overset{Fr_1}{\longrightarrow} \\ \overset{Fc}{\longleftarrow} \\ \underset{Fr_2}{\longrightarrow} \end{array} \; FC \overset{q}{\twoheadrightarrow} \bar{F}$$

is a reflexive coequaliser. Therefore

$$\begin{array}{ccccc} FR & \underset{Fr_2}{\overset{Fr_1}{\rightrightarrows}} & FC & \overset{q}{\twoheadrightarrow} & \overline{F} \\ {\scriptstyle Fsr_1}\downarrow & & \downarrow{\scriptstyle Fs} & & \downarrow{\scriptstyle \overline{s}} \\ FM & \underset{1_{FM}}{\overset{1_{FM}}{\rightrightarrows}} & FM & \overset{1_{FM}}{\longrightarrow} & FM \end{array}$$

is a reflexive coequaliser in $\mathcal{E}/M$. The same is true if we replace s with t and $\bar{s}$ with $\bar{t}$. Now since reflexive coequalisers commute with products we deduce that

$$\begin{array}{ccccc} 2_\times FR & \underset{2_\times Fr_2}{\overset{2_\times Fr_1}{\rightrightarrows}} & 2_\times FC & \overset{2_\times q}{\twoheadrightarrow} & 2_\times \overline{F} \\ \downarrow{\scriptstyle \pi_2 Fsr_1} & & \downarrow{\scriptstyle \pi_2 Fs} & & \downarrow{\scriptstyle \pi_2 \overline{s}} \\ FM & \underset{1_{FM}}{\overset{1_{FM}}{\rightrightarrows}} & FM & \overset{1_{FM}}{\longrightarrow} & FM \end{array}$$

is a reflexive coequaliser in $\mathcal{E}/M$ and finally that the top line is a reflexive coequaliser in $\mathcal{E}$. The required result follows by an easy induction. □

Lemma 3.1.7. *Let F be a precategory in $\mathcal{E}$ and $\overline{F}$ be its quotient reflexive graph. Then the arrow $\overline{m}$ defined by the factorisation of m through the coequaliser*

$$\begin{array}{ccccc} 2_\times FR & \overset{2_\times Fr_1}{\underset{2_\times Fr_2}{\rightrightarrows}} & 2_\times FC & \xrightarrow{2_\times q} & 2_\times \overline{F} \\ \downarrow{\scriptstyle n} & & \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle \overline{m}} \\ FR & \overset{Fr_1}{\underset{Fr_2}{\rightrightarrows}} & FC & \xrightarrow{q} & \overline{F} \end{array}$$

is a well defined composition on $\overline{F}$. When $\overline{F}$ is equipped with this composition we call it the quotient category of the precategory F.

Proof. The associativity and unit axioms are deduced easily from the corresponding diagrams in Definition 3.1.4. The composition $\overline{m}$ is associative because

$$\begin{array}{ccccc} & & 3_\times FC & \xrightarrow{3_\times q} & 3_\times \overline{F} \\ & \swarrow{\scriptstyle Fa} & {\scriptstyle Fm(Fm\times 1)}\downarrow\downarrow{\scriptstyle Fm(1\times Fm)} & & {\scriptstyle \overline{m}(1\times \overline{m})}\downarrow\downarrow{\scriptstyle \overline{m}(\overline{m}\times 1)} \\ FR & \overset{Fr_1}{\underset{Fr_2}{\rightrightarrows}} & FC & \xrightarrow{q} & \overline{F} \end{array}$$

is serially commutative and $3_\times q$ is an epimorphism. The unit axioms hold because

$$\begin{array}{ccccc} & & FC & \xrightarrow{q} & \overline{F} \\ & \swarrow{\scriptstyle F\lambda} & {\scriptstyle Fm(1\times Fe)}\downarrow\downarrow{\scriptstyle 1_{FC}} & & {\scriptstyle \overline{m}(1\times \overline{e})}\downarrow\downarrow{\scriptstyle 1_{\overline{F}}} \\ FR & \overset{Fr_1}{\underset{Fr_2}{\rightrightarrows}} & FC & \xrightarrow{q} & \overline{F} \end{array}$$

and

$$\begin{array}{ccccc} & & FC & \xrightarrow{q} & \overline{F} \\ & \swarrow{\scriptstyle F\rho} & {\scriptstyle Fm(Fe\times 1)}\downarrow\downarrow{\scriptstyle 1_{FC}} & & {\scriptstyle \overline{m}(\overline{e}\times 1)}\downarrow\downarrow{\scriptstyle 1_{\overline{F}}} \\ FR & \overset{Fr_1}{\underset{Fr_2}{\rightrightarrows}} & FC & \xrightarrow{q} & \bar{F} \end{array}$$

are serially commutative and q is an epimorphism. □

Corollary 3.1.8. *The function $\overline{(-)}$ extends to a functor*

$$PreCat(\mathcal{E}) \to Cat(\mathcal{E})$$

Proof. Let $F, K : Pre \to \mathcal{E}$ be precategories in $\mathcal{E}$ and $f : F \Rightarrow K$ be a natural transformation between them. We will show that $(\bar{f}, f_M)$ is an internal functor where $\bar{f}$ is defined as the following factorisation

$$\begin{array}{ccccc}
FR & \overset{Fr_1}{\underset{Fr_2}{\rightrightarrows}} & FC & \xrightarrow{q} & \overline{F} \\
\downarrow{\scriptstyle f_R} & & \downarrow{\scriptstyle f_C} & & \vdots{\scriptstyle \bar{f}} \\
KR & \overset{Kr_1}{\underset{Kr_2}{\rightrightarrows}} & KC & \xrightarrow{q} & \overline{K}
\end{array}$$

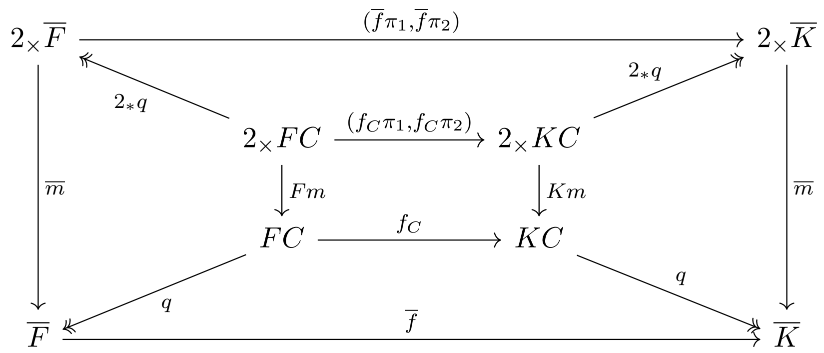
Since f is a natural transformation in particular we have that

$$\begin{array}{ccc}
F(2_\times C) & \xrightarrow{(f_C\pi_1, f_C\pi_2)} & K(2_\times C) \\
\downarrow{\scriptstyle Fm} & & \downarrow{\scriptstyle Km} \\
FC & \xrightarrow{f_C} & KC
\end{array}$$

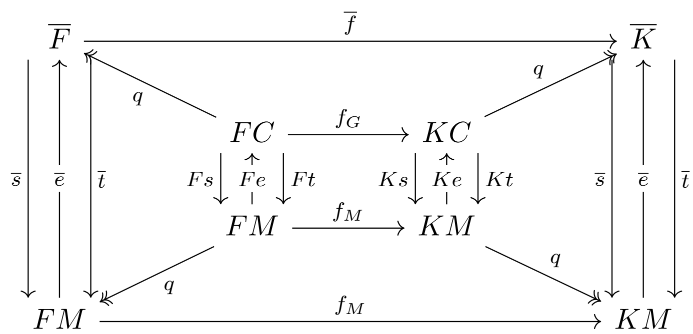
and

$$\begin{array}{ccc}
FC & \xrightarrow{f_C} & KC \\
{\scriptstyle Fs}\downarrow \; \uparrow{\scriptstyle Fe} \; \downarrow{\scriptstyle Ft} & & {\scriptstyle Ks}\downarrow \; \uparrow{\scriptstyle Ke} \; \downarrow{\scriptstyle Kt} \\
FM & \xrightarrow{f_M} & KM
\end{array}$$

are (serially) commutative. But this means that the inner squares of



and



are (serially) commutative. But now we see that the outer squares commute due to the top left diagonal arrow being an epimorphism. Hence $(\overline{f}, f_M) : \overline{F} \to \overline{K}$ is an internal functor. The functor preserves composition because of the uniqueness of factorisation through the colimit defining $\overline{F}$. □

Remark 3.1.9. The functor $\overline{(-)}$ is left adjoint to the natural inclusion

$$Cat(\mathcal{E}) \xrightarrow{y} PreCat(\mathcal{E})$$

which will be described in more detail in Definition 3.4.5.

Remark 3.1.10. It is routine to specialise the precategory construction to a 'pregroupoid' construction. In the remainder of this subsection we sketch the details.

Definition 3.1.11. The category Pre' is generated from the category Pre by adjoining arrows i_C, i_R, I_1 and I_2 such that the equations $si_C = t$ and $ti_C = s$ hold and the diagrams

$$\begin{array}{ccc} R & \overset{r_1}{\underset{r_2}{\rightrightarrows}} & C \\ \downarrow{\scriptstyle i_R} & & \downarrow{\scriptstyle i_C} \\ R & \overset{r_1}{\underset{r_2}{\rightrightarrows}} & C \end{array}$$

$$\begin{array}{ccc} & & C \\ & \swarrow{\scriptstyle I_1} & {\scriptstyle es}\downarrow\downarrow{\scriptstyle m(i_C\times 1)} \\ R & \overset{r_1}{\underset{r_2}{\rightrightarrows}} & C \end{array} \quad \text{and} \quad \begin{array}{ccc} & & C \\ & \swarrow{\scriptstyle I_2} & {\scriptstyle et}\downarrow\downarrow{\scriptstyle m(1\times i_C)} \\ R & \overset{r_1}{\underset{r_2}{\rightrightarrows}} & C \end{array}$$

are serially commutative.

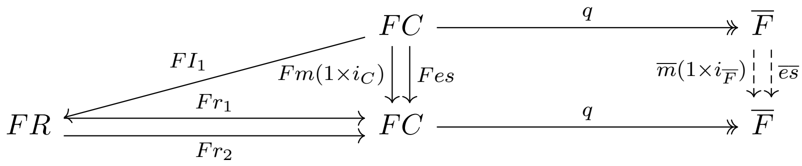
Definition 3.1.12. A pregroupoid in $\mathcal{E}$ is a finite limit preserving functor $Pre \to \mathcal{E}$. The category of pregroupoids in $\mathcal{E}$ is the category $[Pre, \mathcal{E}]_{lex}$ and will sometimes be written $PreGpd(\mathcal{E})$. Here the subscript lex indicates that we consider only functors that preserve finite limits.

Lemma 3.1.13. *Let F be a pregroupoid in $\mathcal{E}$ and $\overline{F}$ be the quotient category of the underlying precategory of F. Then using the arrow $i_{\overline{F}}$ induced by the factorisation*

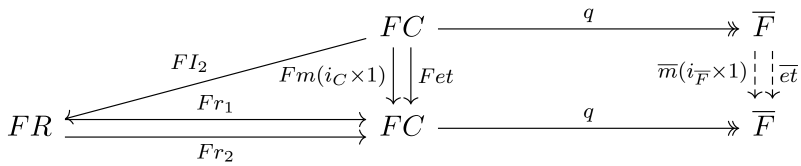
$$\begin{array}{ccccc} FR & \overset{Fr_1}{\underset{Fr_2}{\rightrightarrows}} & FC & \overset{q}{\twoheadrightarrow} & \overline{F} \\ \downarrow{\scriptstyle Fi_R} & & \downarrow{\scriptstyle Fi_C} & & \downarrow{\scriptstyle i_{\overline{F}}} \\ FR & \overset{Fr_1}{\underset{Fr_2}{\rightrightarrows}} & FC & \overset{q}{\twoheadrightarrow} & \overline{F} \end{array}$$

as the inverse map makes $\overline{F}$ into a groupoid in $\mathcal{E}$.

Proof. Given the work in Corollary 3.1.8, it suffices to observe that the diagrams



and



are serially commutative. □

Now the following proposition follows immediately.

Proposition 3.1.14. *The function $\overline{(-)}$ extends to a functor $PreGpd(\mathcal{E}) \to Gpd(\mathcal{E})$.*

Remark 3.1.15. The functor $\overline{(-)}$ is left adjoint to the natural inclusion

$$Gpd(\mathcal{E}) \xrightarrow{y} PreGpd(\mathcal{E})$$

3.2 Concatenation

In order to define the path and Weinstein precategories that follow it will be necessary to define a way of concatenating paths. We will do this by defining an arrow $\mu : \mathbb{I} \to 2_*\mathbb{I}$ where $2_*\mathbb{I}$ is the pushout $\mathbb{I}\ {}_1{+}_0\ \mathbb{I}$ in $Cat(\mathcal{E})$. We think of $2_*\mathbb{I}$ as the representing category for composable paths and so for our concatenation μ to make sense we require that $\mu(l) = \iota_2 l \circ \iota_1 l$ where l is the long arrow $0 \to 1$ in $\mathbb{I}$. First we show that the pushout $2_*\mathbb{I}$ in $Cat(\mathcal{E})$ is in fact a preorder and hence that any arrow $\mathbb{I} \to 2_*\mathbb{I}$ is determined by its object map $I \to 2_*I$ where 2_*I is the pushout $I\ {}_1{+}_0\ I$. Then we simplify the problem of mapping into this pushout by carving 2_*I out as a subobject of the product $I \times I$. Finally it is easy to construct an appropriate smooth function $I \to I \times I$ that factors through this subobject.

Notation 3.2.1. For a category $\mathbb{C}$ in $\mathcal{E}$ and a global element $* : 1 \to M$ we write $2_*\mathbb{C}$ for the pushout $\mathbb{C} \;{}_*{+}_*\; \mathbb{C}$ in $Cat(\mathcal{E})$. In the special case when $\mathbb{C} = \mathbb{I}$ we modify this notation slightly so that it is easier to write iterated pushouts. Therefore we write $n_*\mathbb{I}$ for the n-fold pushout $\mathbb{I} \;{}_1{+}_0 \cdots {}_1{+}_0\; \mathbb{I}$.

Let $\mathcal{C}$ be one of the categories $\mathcal{C}_W$, $\mathcal{C}_{jet}$, $\mathcal{C}_{fp}$ or $\mathcal{C}_{germ}$ defined in Section 1.1 and $\mathcal{E}$ be the well-adapted model that is constructed $\mathcal{C}$ by taking sheaves with respect to the Dubuc coverage. It will be convenient to have a way of relating $\mathcal{E}$ to the category of presheaves on $\mathcal{C}$ so we recall Theorem 1 from Section III.5 of [23]:

Theorem 3.2.2. *The inclusion functor $j : \mathcal{E} \rightarrowtail [\mathcal{C}^{op}, Set]$ has a left adjoint*

$$a : [\mathcal{C}^{op}, Set] \to \mathcal{E}$$

called the associated sheaf functor. Moreover, this functor a commutes with finite limits.

Corollary 3.2.3. *There is an adjunction*

$$[\mathcal{C}^{op}, Cat] \underset{J}{\overset{A}{\underset{\hookleftarrow}{\overset{\to}{\bot}}}} Cat(\mathcal{E})$$

such that A preserves finite limits as well as all colimits.

Lemma 3.2.4. *Let $\mathbb{C}$ be a category in $\mathcal{E}$. Then $\mathbb{C}$ is a preorder iff $J\mathbb{C}$ is a preorder. Let $\mathbb{D}$ be in $[\mathbb{C}^{op}, Cat]$. Then $\mathbb{D}$ is a preorder iff $A\mathbb{D}$ is a preorder.*

Proof. Both j and a preserve products and take monomorphisms to monomorphisms. □

Lemma 3.2.5. *For any preorder $\mathbb{C}$ in $Cat(\mathcal{E})$ defined by $C \rightarrowtail M \times M$ and global element $* : 1 \to M$ we have that $2_*\mathbb{C}$ is a preorder.*

Proof. By Lemma 3.2.4 the hypothesis that $\mathbb{C}$ is a preorder implies that $J\mathbb{C}$ is a preorder. Now $2_*(J\mathbb{C})$ is a preorder in $[\mathcal{C}^{op}, Cat]$ because pushouts are computed componentwise and the corresponding result is easily checked to hold in Cat. Hence by Lemma 3.2.4 and the fact that A preserves colimits we see that

$$A(2_*J\mathbb{C}) \cong 2_*\mathbb{C}$$

is a preorder as required. □

In order to ensure that the concatenation operation is sufficiently smooth we will need a smooth step function $\mathbb{R} \to \mathbb{R}$ which takes the value 0 for $x < 0$ and 1 for $x > 1$.

Definition 3.2.6. Let $\int$, exp denote the classical integral and exponential function respectively. Then we define the smooth function

$$\delta' : \mathbb{R} \to \mathbb{R}$$
$$x \mapsto \begin{cases} \frac{\int_0^x exp(\frac{-t}{1-t})dt}{\int_0^1 exp(\frac{-t}{1-t})dt} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

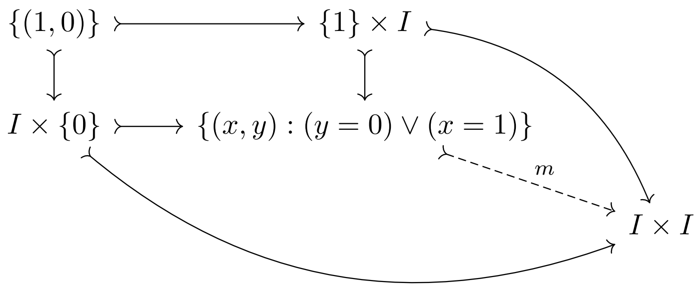
Remark 3.2.7. Applying the full and faithful embedding $\iota : Man \to \mathcal{E}$ to δ' gives us an arrow $\delta : R \to R$ in $\mathcal{E}$.

Lemma 3.2.8. *Let $\nu : I \to I \times I$ be an arrow in $\mathcal{E}$. Then ν induces an arrow $I \to 2_* I$ iff there exists a cover*

$$A \xrightarrow{a} I \xleftarrow{b} B$$

of I such that $\nu \circ a$ factors through $(1_I, \{0\})$ and $\nu \circ b$ factors through $(\{1\}, 1_I)$.

Proof. First observe that in the diagram



the outer square is a pullback and the top left square is a pushout-pullback. Then using the evident isomorphisms $I \times \{0\} \cong I \cong \{1\} \times I$ and $\{(1,0)\} \cong 1$ we find that $\{(x,y) : (y = 0) \vee (x = 1)\} \cong I \; {}_0{+}_1 \; I$.

Now the result follows immediately from the characterisation of disjunction in the Kripke-Joyal semantics of the topos. See for example Theorem 1 (ii) in Part VI.6 of [23]. □

In the following result let $\leq_{2_*I}$ be the order relation on 2_*I corresponding to $2_*\mathbb{I}$.

Corollary 3.2.9. *Let $a : I \to I$ be defined by $u \mapsto \frac{3}{4}u$ and $b : I \to I$ be defined by $u \mapsto \frac{1}{4} + \frac{3}{4}u$. Let $\nu : I \to I \times I$ be induced by the pair $(\delta(8x), \delta(8x-7))$. Then there exists a $\mu : I \to 2_*I$ such that $\nu = m \circ \mu$, the proposition $x \leq x' \implies \mu(x) \leq_{2_*I} \mu(x')$ holds and $\mu(0) = 0$ and $\mu(1) = 1$.*

Proof. We need to check the conditions of Lemma 3.2.8. Clearly a and b are jointly epimorphic. Also $\nu \circ a$ always has second component 0 because $\delta(8x-7)$ is 0 for $x < \frac{7}{8}$. Finally $\nu \circ b$ always has first component 1 because $\delta(8x)$ is 1 for $x > \frac{1}{8}$. Since δ is monotone the inequality $x \leq x'$ implies that both $\delta(8x) \leq \delta(8x')$ and $\delta(8x-7) \leq \delta(8x'-7)$. Hence $x \leq x' < \frac{3}{4}$ implies that $\mu(x) \leq_{2_*I} \mu(x')$ and $\frac{1}{4} < x \leq x'$ implies that $\mu(x) \leq_{2_*I} \mu(x')$ as required. $\square$

Corollary 3.2.10. *The arrow μ induces a functor $\mu : \mathbb{I} \to 2_*\mathbb{I}$ such that $\mu(l) = \iota_2 l \circ \iota_1 l$ where l is the long arrow $0 \to 1$ in $\mathbb{I}$.*

Remark 3.2.11. If we instead want to produce a groupoid homomorphism $\mu : \nabla I \to 2_*\nabla I$ we use the same arguments as above but replace $Cat(\mathcal{E})$ by $Gpd(\mathcal{E})$, L by C_I (and hence $\mathbb{I}$ by ∇I). An easy adaptation of Lemma 3.2.5 shows that $2_*\nabla I \cong \nabla(2_*I)$ and so in this situation it is not necessary to consider any order relation and it will again suffice to give an arrow $I \to 2_*I$ as furnished by Corollary 3.2.9.

Corollary 3.2.12. *There exists a groupoid homomorphism $\mu : \nabla I \to 2_*\nabla I$ such that $\mu(l) = \iota_2 l \circ \iota_1 l$.*

3.3 Connectedness of Categories

Recall that in classical Lie theory we have an equivalence of categories between the category of Lie algebras and the category of simply connected Lie groups. To see why we must impose the condition of path connectivity consider a Lie group G which we can decompose into two connected components $G = G_1 \amalg G_2$. Without loss of generality we may suppose that $e \in G_1$ which would mean that the Lie algebras of G and G_1 are identical. But there is no isomorphism between

G and G_1 in *LieGp*. We can make a similar argument to see that it is essential to impose the condition that the Lie groups are simply connected. Consider a Lie group H which is not simply connected and hence is not isomorphic to its universal covering space UH in *LieGp*. However we now observe that the covering projection $p : UH \to H$ is a local diffeomorphism and hence the Lie algebras of H and UH are the same.

In the situation where the base space is not a singleton and we allow non-invertible arrows we encounter precisely the same problem. (Although in this case solving this problem is not enough to ensure that we get an equivalence of categories.) Therefore in this section we will identify conditions on categories in $\mathcal{E}$ that will be the counterparts of imposing the conditions of path and simply connectedness on Lie groups. Then we show that the representing object for n-fold composition $n_*\mathbb{I}$ is simply connected. This result will be used to obtain the associativity and unit arrows of the path precategory formed in the following section.

Recall that a topological space X is path connected iff it satisfies the following weak lifting property: for all pairs of points $(x, y) \in X \times X$ there exists a path γ such that the following diagram

$$\begin{array}{ccc} 1+1 & \xrightarrow{(x,y)} & X \\ {\scriptstyle (0,1)}\downarrow & \nearrow_{\gamma} & \\ I & & \end{array}$$

commutes. We can rephrase this as the statement: the arrow

$$X^I \xrightarrow{X^{(0,1)}} X^{1+1}$$

is an epimorphism in *Set*. Recall further that a Lie groupoid $\mathbb{G}$ is s-path connected iff each of the source fibres $s^{-1}M$ are path connected in *Man*. Using Lemma 5.1.2 we see that a Lie groupoid is s-path connected iff the arrow

$$\mathbb{G}^{\nabla I} \xrightarrow{\mathbb{G}^l} \mathbb{G}^{\mathbf{2}}$$

is an epimorphism in *Set*. The only modification to this definition that we will make is to insist that the arrow $\mathbb{G}^l$ is not only an epimorphism in *Set* but also an epimorphism in $\mathcal{E}$. Besides being a more natural condition when working internally in $\mathcal{E}$ this will be required for the proof of Lie's second theorem; we eventually need to produce an arrow between two objects in $\mathcal{E}$ not just one between the sets of global sections.

Definition 3.3.1. A object $X \in \mathcal{E}$ is path connected iff the arrow

$$X^I \xrightarrow{X^{(0,1)}} X^{1+1}$$

is an epimorphism in $\mathcal{E}$.

Definition 3.3.2. A groupoid $\mathbb{G}$ in $\mathcal{E}$ is path connected iff the arrow

$$\mathbb{G}^{\nabla I} \xrightarrow{\mathbb{G}^l} \mathbb{G}^{\mathbf{2}}$$

is an epimorphism in $\mathcal{E}$.

The next Lemma relates path connected objects of $\mathcal{E}$ and path connected groupoids in $Gpd(\mathcal{E})$.

Lemma 3.3.3. *If X is path connected in $\mathcal{E}$ then ∇X is path connected in $Gpd(\mathcal{E})$.*

Proof. The result follows immediately from Lemma 5.1.2 and the isomorphism $\nabla B^{\nabla A} \cong B^A$. □

For our definition of path connected category we will take the definition of path connected groupoid and replace ∇I with $\mathbb{I}$.

Definition 3.3.4. A category $\mathbb{C}$ in $\mathcal{E}$ is path connected iff the arrow

$$\mathbb{C}^{\mathbb{I}} \xrightarrow{\mathbb{C}^l} \mathbb{C}^{\mathbf{2}}$$

is an epimorphism in $\mathcal{E}$.

In order to write down the definition of simply connected category we will need to define two additional categories: one to be the representing object for homotopies with fixed endpoints and the other to be its boundary.

Definition 3.3.5. The category O in $\mathcal{E}$ is the pushout of $l : \mathbf{2} \to \mathbb{I}$ along itself:

$$\begin{array}{ccc} \mathbf{2} & \xrightarrow{l} & \mathbb{I} \\ \downarrow{\scriptstyle l} & & \downarrow{\scriptstyle \iota_1} \\ \mathbb{I} & \xrightarrow{\iota_2} & O \end{array}$$

in $Cat(\mathcal{E})$. Note that the space of objects Ω of O is the pushout

$$\begin{array}{ccc} 1+1 & \xrightarrow{(0,1)} & I \\ \downarrow{\scriptstyle (0,1)} & & \downarrow \\ I & \longrightarrow & \Omega \end{array}$$

in $\mathcal{E}$.

Remark 3.3.6. The space Ω is representable in $\mathcal{E}$. To see this consider the closed subset Ω_2 of R^2 that consists of all the pairs $(x, y) \in I^2$ such that the proposition

$$(y = sin(x)) \vee (y = -sin(x))$$

holds. Since the curves $y = sin(x)$ and $y = -sin(x)$ are transversal at $(0, 0)$ and $(1, 0)$ we see that giving a smooth function $f : \Omega_2 \to \mathbb{R}$ is the same as giving a pair of smooth functions $(f_1, f_2) : I \to \mathbb{R}$ such that $f_1(0) = f_2(0)$ and $f_1(1) = f_2(1)$. Hence

$$\begin{array}{ccc} [2, m^0_{1+1}] & \longrightarrow & [2, m^0_I] \\ \downarrow & & \downarrow \\ [2, m^0_I] & \longrightarrow & [2, m^0_{\Omega_2}] \end{array}$$

is a pushout in $\mathcal{C}_{pt}$ where m^0_X is the ideal of all smooth functions vanishing on X. Therefore Ω is represented by $[2, m^0_{\Omega_2}]$ in $\mathcal{E}$.

Definition 3.3.7. For any category $\mathbb{C}$ in $\mathcal{E}$ and arrow $f : \mathbf{2} \to \mathbb{C}$ the category $\mathbb{O}^f_{\mathbb{C}}$ is the pushout

$$\begin{array}{ccc} \mathbf{2} \times \mathbb{C} & \xrightarrow{f \times 1_{\mathbb{I}}} & \mathbb{C} \times \mathbb{C} \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle q^f_{\mathbb{C}}} \\ \mathbf{2} & \longrightarrow & \mathbb{O}^f_{\mathbb{C}} \end{array}$$

in $Cat(\mathcal{E})$. In the case that $\mathbb{C} = \mathbb{I}$ and $f = l : \mathbf{2} \to \mathbb{I}$ we will write simply $\mathbb{O}$ for $\mathbb{O}^l_{\mathbb{I}}$. Note that the space of objects of $\mathbb{O}$ is given by the pushout

$$\begin{array}{ccc} I + I & \xrightarrow{((0,1_I),(1,1_I))} & I \times I \\ \downarrow{\scriptstyle (u_1,u_2)} & & \downarrow{\scriptstyle q} \\ 1+1 & \longrightarrow & B \end{array}$$

where u_1 and u_2 are the coproduct inclusions and in turn B is isomorphic to the colimit of the diagram

$$1 \xleftarrow{!} I \xrightarrow{(0,1_I)} I \times I \xleftarrow{(1,1_I)} I \xrightarrow{!} 1$$

in $\mathcal{E}$.

Remark 3.3.8. Note that we could have defined the category O as the pushout

$$\begin{array}{ccc} (1+1)\times \mathbf{2} & \xrightarrow{(1+1)\times l} & (1+1)\times \mathbb{I} \\ \downarrow{\scriptstyle \pi_2} & & \downarrow \\ \mathbf{2} & \longrightarrow & O \end{array}$$

and so the arrow $(0,1) : (1+1) \to I$ induces the boundary inclusion $\iota : O \to \mathbb{O}$.

Before we define a simply connected category we prove that when the category $\mathbb{C}$ is a preorder then the category $\mathbb{O}_{\mathbb{C}}$ is a preorder also.

Lemma 3.3.9. *The category $\mathbb{O}_{\mathbb{C}}$ is a preorder when the category $\mathbb{C}$ is a preorder.*

Proof. Let j, a, J and A be the functors of Theorem 3.2.2 and Corollary 3.2.3. Since j preserves limits the category $J\mathbb{C}$ is in fact a preorder and $J(\mathbb{C} \times \mathbb{C}) \cong J\mathbb{C} \times J\mathbb{C}$. Then it is easy to see that in $[\mathcal{C}^{op}, Cat]$ the pushout $\mathbf{2}_{+\mathbf{2}\times J\mathbb{C}}(J\mathbb{C} \times J\mathbb{C})$ is a preorder. Since A preserves finite limits and all colimits by Lemma 3.2.4 we have that

$$A(\mathbf{2}_{+\mathbf{2}\times J\mathbb{C}}(J\mathbb{C} \times J\mathbb{C})) \cong \mathbf{2}_{+\mathbf{2}\times J\mathbb{C}}(J\mathbb{C} \times J\mathbb{C}) \cong \mathbb{O}_{\mathbb{C}}$$

is a preorder as required. □

In Lemma 5.1.2 we see that maps $\nabla B \to \mathbb{G}$ in $Gpd(\mathcal{E})$ are the same as maps $\psi : B \to G$ in $\mathcal{E}$ that are source constant and have $\psi(0,0) = es\psi(0,0)$. Hence ∇B is the representing object for source constant homotopies in $\mathbb{G}$ with fixed endpoints. This motivates the following definition which generalises the definition of source simply connected Lie groupoid found in classical multi-object Lie theory by replacing ∇B with $\mathbb{O}$ and Set with $\mathcal{E}$.

Definition 3.3.10. A category $\mathbb{C}$ is simply connected iff $\mathbb{C}$ is path connected and the arrow

$$\mathbb{C}^{\mathbb{O}} \xrightarrow{\mathbb{C}^{\iota}} \mathbb{C}^{O}$$

is an epimorphism in $\mathcal{E}$.

Lemma 3.3.11. *Let $\mathbb{C}$ be a simply connected category via the epimorphisms $m : \mathbb{C}^{\mathbb{O}} \twoheadrightarrow \mathbb{C}^{O}$ and $a : \mathbb{C}^{\mathbb{I}} \twoheadrightarrow \mathbb{C}^{\mathbf{2}}$. Let $\mathbb{D}$ be a simply connected category via the epimorphisms $n : \mathbb{D}^{\mathbb{O}} \twoheadrightarrow \mathbb{D}^{O}$ and $b : \mathbb{D}^{\mathbb{I}} \twoheadrightarrow \mathbb{D}^{\mathbf{2}}$. Then $\mathbb{C} \times \mathbb{D}$ is simply connected.*

Proof. The arrows

$$(\mathbb{C} \times \mathbb{D})^{\mathbb{I}} \cong \mathbb{C}^{\mathbb{I}} \times \mathbb{D}^{\mathbb{I}} \xrightarrow{a \times b} \mathbb{C}^{\mathbf{2}} \times \mathbb{D}^{\mathbf{2}} \cong (\mathbb{C} \times \mathbb{D})^{\mathbf{2}}$$

and

$$(\mathbb{C} \times \mathbb{D})^{\mathbb{O}} \cong \mathbb{C}^{\mathbb{O}} \times \mathbb{D}^{\mathbb{O}} \xrightarrow{m \times n} \mathbb{C}^{O} \times \mathbb{D}^{O} \cong (\mathbb{C} \times \mathbb{D})^{O}$$

are epimorphisms exhibiting $\mathbb{C} \times \mathbb{D}$ as simply connected. □

Lemma 3.3.12. *The category $n_*\mathbb{I}$ is path connected.*

Proof. We need to show that the arrow

$$(n_*\mathbb{I})^{\mathbb{I}} \xrightarrow{(n_*\mathbb{I})^{\iota}} (n_*\mathbb{I})^{\mathbf{2}}$$

is an epimorphism in $\mathcal{E}$. By iterating Lemma 3.2.5 we see that $n_*\mathbb{I}$ is a preorder and so the arrow $(s, t) : (n_*\mathbb{I})^{\mathbf{2}} \rightarrowtail (n_*I)^2$ is a monomorphism. Hence all internal functors into $n_*\mathbb{I}$ are completely determined by the object map. Let $\leq_{n_*I}$ be the order induced on n_*I by $n_*\mathbb{I}$. This means that

$$(n_*\mathbb{I})^{\mathbb{I}} = \{\phi \in (n_*I)^I : \forall a, a' \in I.\ a \leq a' \implies \phi(a) \leq_{n_*I} \phi(a')\}$$

and

$$(n_*\mathbb{I})^{\mathbf{2}} = \{(x, y) \in (n_*I)^2 : x \leq_{n_*I} y\}$$

Therefore it will suffice to show that the proposition

$$\forall (x, y) \in (n_*I)^2.\ \exists \psi \in (n_*I)^I.\ (\psi(0) = x) \wedge (\psi(1) = y)$$

holds in the internal logic of $\mathcal{E}$ and that whenever $x \leq_{n_*I} y$ we can choose ψ to lie in $(n_*\mathbb{I})^{\mathbb{I}}$. So let $(x, y) \in (n_*I)^2$ with x in the ith summand and y in the

jth summand for $i, j \in \{1, ..., n\}$ and $i \leq j$. First we consider the case $i = j$. Then we can choose $\psi = \iota_i \psi'$ where $\psi' : I \to I$ is defined by

$$\psi'(a) = (1 - a)x_i + ay_i$$

where $\iota_i x_i = x$ and $\iota_i y_i = y$. Second we consider the case $i < j$. Then we choose ψ to be the concatenation (using the μ of Corollary 3.2.9) of the paths $\iota_i\psi_i$, $\iota_{i+1}\psi_{i+1}$,...,$\iota_j\psi_j$ where ψ_i,...,ψ_j are defined as follows.

$$\psi_i(a) = (1 - a)x_i + a \ , \ \psi_j(a) = ay_j$$

and $\psi_k = 1_I$ for all $k \in \{i + 1, ..., j - 1\}$ where $\iota_i x_i = x$ and $\iota_j y_j = y$. By construction the ψ produced in both cases does indeed satisfy the proposition

$$\forall a, a' \in I.\ a \leq a' \implies \phi(a) \leq_{n_*I} \phi(a')$$

provided that $x \leq_{n_*I} y$. □

Lemma 3.3.13. *The category $n_*\mathbb{I}$ is simply connected.*

Proof. We will prove the case $n = 3$. The general case is more difficult only in terms of notation. Given Lemma 3.3.12 it suffices to show that the arrow

$$(3_*\mathbb{I})^{\mathbb{O}} \xrightarrow{(3_*\mathbb{I})^{\iota}} (3_*\mathbb{I})^{O}$$

is an epimorphism in $\mathcal{E}$. First we note that because $3_*\mathbb{I}$ is a preorder the arrow $(s, t) : (3_*\mathbb{I})^{\mathbf{2}} \rightarrowtail (3_*I)^2$ is a monomorphism and hence all internal functors into $3_*\mathbb{I}$ are completely determined by the object map. Let $\leq_{3_*I}$ be the order induced on 3_*I by $3_*\mathbb{I}$. Then

$$(3_*\mathbb{I})^{\mathbb{O}} = \{\psi \in (3_*I)^B :\ \forall a, a' \in B.\ a_1 \leq a'_1 \implies \psi(a) \leq_{3_*I} \psi(a')\}$$

and

$$(3_*\mathbb{I})^{O} = \{\phi \in (3_*I)^{\Omega} :\ \forall a, a' \in \Omega.\ a_1 \leq a'_1 \implies \phi(a) \leq_{3_*I} \phi(a')\}$$

Second we note that we can carve out 3_*I as a subobject of I^3:

$$3_*I = \{(a, b, c) : (b = c = 0) \vee ((a = 1) \wedge (c = 0)) \vee (a = b = 1)\} \xrightarrow{\nu} I^3$$

Therefore it will suffice to show that the proposition

$$\forall \phi \in (I^3)^\Omega.\ \exists \psi \in (I^3)^B.\ \psi \circ \iota = \phi$$

holds in the internal logic of $\mathcal{E}$ and that whenever ϕ factors through 3_*I and preserves the order, we can choose a filler ψ that factors through 3_*I and preserves the order also.

So let $\phi \in (I^3)^\Omega$ such that

$$(\phi_2 = \phi_3 = 0) \vee ((\phi_1 = 1) \wedge (\phi_3 = 0)) \vee (\phi_1 = \phi_2 = 1)$$

and

$$\forall x, x' \in \Omega.\ x_1 \leq x'_1 \implies \phi(x) \leq_{3_*I} \phi(x')$$

hold. Then define $\psi \in (I^3)^{I \times I}$ by

$$\psi_i(x_1, x_2) = x_2 \delta_i(x_1) + (1 - x_2)\gamma_i(x_1)$$

for $i \in \{1, 2, 3\}$ and where $\phi_i = (\delta_i, \gamma_i)$. Now ψ satisfies the equations $F(0, x_2) = F(0, 0)$ and $F(1, x_2) = F(1, 0)$ for all x_2 because $\delta_i(0) = \gamma_i(0)$ and $\delta_i(1) = \gamma_i(1)$. Hence ψ induces a well defined arrow out of B. In addition $\phi_i = 0$ implies that $\delta_i = 0 = \gamma_i$ so the proposition

$$(\psi_2 = \psi_3 = 0) \vee ((\psi_1 = 1) \wedge (\psi_3 = 0)) \vee (\psi_1 = \psi_2 = 1)$$

holds. Finally $\phi(x_1, x_2) \leq_{3_*I} \phi(x'_1, x'_2)$ and $\phi(x_1, -x_2) \leq_{3_*I} \phi(x'_1, -x'_2)$ together imply that $\delta(x_1) \leq \delta(x'_1)$ and $\gamma(x_1) \leq \gamma(x'_1)$ and so the proposition

$$(\forall a, a' \in B) a_1 \leq a'_1 \implies \psi(a) \leq_{3_*I} \psi(a')$$

holds also as required. □

Remark 3.3.14. If we replace $Cat(\mathcal{E})$ with $Gpd(\mathcal{E})$ and $\mathbb{I}$ with ∇I the theory goes through in the same manner except that we do not need to consider the order relations. Thus we obtain definitions of path and simply connected groupoids qua groupoids and the following result.

Proposition 3.3.15. *The groupoid $n_*\nabla I$ is simply connected as a groupoid.*

3.4 The Path Category and Precategory

In the proof of Lie's second theorem in Chapter 4 our strategy will be to transfer a certain lifting problem into the category of precategories. The left arrow of the square defining this lifting problem will be obtained using the path precategory construction given in this section. It turns out that the condition that our categories are simply connected will only be used to transfer this lifting problem; the rest of the proof is integration.

First we see how to form the path precategory $p(\mathbb{C})$ of a category $\mathbb{C}$ in $\mathcal{E}$. We will do this by specifying an appropriate finite limit preserving functor $\mathcal{I} : Pre \to Cat(\mathcal{E})^{op}$. Since the $\mathcal{E}$-valued hom in $Cat(\mathcal{E})$ gives us a functor

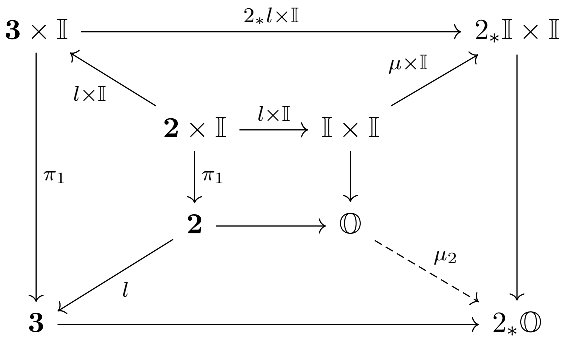
$$[Pre, Cat(\mathcal{E})^{op}]_{lex} \times Cat(\mathcal{E}) \to [Pre, \mathcal{E}]_{lex}$$

we can then induce a functor $p : Cat(\mathcal{E}) \to PreCat(\mathcal{E})$ from this data. The first result from this section that we will require in the proof of Lie's second theorem is that the functor p is faithful on arrows whose codomain is path connected.

We then define the path category $P(\mathbb{C})$ of a category $\mathbb{C}$ in $\mathcal{E}$ as the quotient category of $p(\mathbb{C})$. For the purpose of comparing the path category $P(\mathbb{C})$ with the category $\mathbb{C}$ itself we form a natural transformation $L : P \Rightarrow 1_{Cat(\mathcal{E})}$. The two other results that we will require in the proof of Lie's second theorem are that the component $L_{\mathbb{C}}$ is an epimorphism if $\mathbb{C}$ is path connected and that $L_{\mathbb{C}}$ is an isomorphism iff $\mathbb{C}$ is simply connected. Finally we remark that all these constructions go through equally well when we specialise from categories to groupoids and give the statements of the corresponding results.

Before we give the definition of path precategory we need to define how to concatenate homotopies.

Definition 3.4.1. The arrow μ_2 which describes the concatenation of homotopies is defined as the following unique factorisation

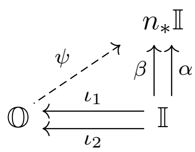


where the inner square and outer squares are pushouts.

Definition 3.4.2. The functor $\mathcal{I} : Pre \to Cat(\mathcal{E})^{op}$ takes

$$R \underset{r_2}{\overset{r_1}{\rightrightarrows}} \xleftarrow{c} C \underset{t}{\overset{s}{\rightrightarrows}} \xleftarrow{e} M \;\mapsto\; \mathbb{O} \underset{\iota_2}{\overset{\iota_1}{\leftleftarrows}} \xrightarrow{\pi} \mathbb{I} \underset{1}{\overset{0}{\leftleftarrows}} \xrightarrow{!} 1,$$

with $\mathcal{I}(m) = \mu : \mathbb{I} \to 2_*\mathbb{I}$ and $\mathcal{I}(n) = \mu_2 : \mathbb{O} \to 2_*\mathbb{O}$. The definitions of $\mathcal{I}(a)$, $\mathcal{I}(\lambda)$ and $\mathcal{I}(\rho)$ are all obtained from the simply connectedness of $n_*\mathbb{I}$ proved in Lemma 3.3.13 which tells us that for all $\alpha, \beta : \mathbb{I} \to n_*\mathbb{I}$ such that $\alpha(l) = \beta(l)$ there exists a ψ such that



is serially commutative.

Definition 3.4.3. The functor $\mathcal{Y} : Pre \to Cat(\mathcal{E})^{op}$ takes

$$R \underset{r_2}{\overset{r_1}{\rightrightarrows}} \xleftarrow{c} C \underset{t}{\overset{s}{\rightrightarrows}} \xleftarrow{e} M \;\mapsto\; \mathbf{2} \underset{1_\mathbf{2}}{\overset{1_\mathbf{2}}{\leftleftarrows}} \xrightarrow{1_\mathbf{2}} \mathbf{2} \underset{t}{\overset{s}{\leftleftarrows}} \xrightarrow{!} 1,$$

with $\mathcal{Y}(m) = \mathcal{Y}(n) = l : \mathbf{2} \to \mathbf{3}$ picking out the long arrow $0 \to 2$. Note that since the groupoid $\mathbf{2}$ satisfies the associativity and unit laws the arrows $\mathcal{Y}(a)$, $\mathcal{Y}(\lambda)$ and $\mathcal{Y}(\rho)$ are trivial to define. For example $\mathcal{Y}(a) = \mathcal{Y}(m(m \times 1)) = \mathcal{Y}(m(1 \times m))$.

Definition 3.4.4. The natural transformation $l' : \mathcal{Y} \Rightarrow \mathcal{I}$ is induced from the diagram homomorphism

$$
\begin{array}{ccccc}
\mathbf{2} & \overset{\iota_1}{\longleftarrow} \ \xrightarrow{\ \pi\ } \ \underset{\iota_2}{\longleftarrow} & \mathbf{2} & \overset{0}{\longleftarrow} \ \xrightarrow{\ !\ } \ \underset{1}{\longleftarrow} & 1 \\
\downarrow{\scriptstyle l\iota_1} & & \downarrow{\scriptstyle l} & & \downarrow{\scriptstyle 1_1} \\
\mathbb{O} & \overset{\iota_1}{\longleftarrow} \ \xrightarrow{\ \pi\ } \ \underset{\iota_2}{\longleftarrow} & \mathbb{I} & \overset{0}{\longleftarrow} \ \xrightarrow{\ !\ } \ \underset{1}{\longleftarrow} & 1.
\end{array}
$$

Remark 3.4.5. Using the $\mathcal{E}$-valued hom in $Cat(\mathcal{E})$ we have a functor

$$[Pre, Cat(\mathcal{E})^{op}]_{lex} \times Cat(\mathcal{E}) \to [Pre, \mathcal{E}]_{lex}$$

Therefore $\mathcal{I}$ and $\mathcal{Y}$ induce functors p and $y : Cat \to PreCat(\mathcal{E})$ respectively and l' induces a natural transformation $l : p \Rightarrow y$. We call the category $p(\mathbb{C})$ the path precategory of $\mathbb{C}$.

Lemma 3.4.6. *The functor $y : Cat \to PreCat(\mathcal{E})$ is full and faithful.*

Proof. Let $\mathbb{C}$ and $\mathbb{D}$ be categories. An arrow $\phi : y(\mathbb{C}) \to y(\mathbb{D})$ in $PreCat(\mathcal{E})$ is a diagram homomorphism ϕ_i in $\mathcal{E}$ such that

$$
\begin{array}{ccccc}
\mathbb{C}^{2_*\mathbf{2}} & \xrightarrow{\ \circ_C\ } & \mathbb{C}^{\mathbf{2}} & \overset{s_C}{\longrightarrow} \ \xleftarrow{\ e_C\ } \ \underset{t_C}{\longrightarrow} & \mathbb{C}^1 \\
{\scriptstyle (\phi_1\pi_1, \phi_1\pi_2)}\downarrow & & \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi_0} \\
\mathbb{D}^{2_*\mathbf{2}} & \xrightarrow{\ \circ_D\ } & \mathbb{D}^{\mathbf{2}} & \overset{s_D}{\longrightarrow} \ \xleftarrow{\ e_D\ } \ \underset{t_D}{\longrightarrow} & \mathbb{D}^1
\end{array}
$$

is serially commutative. But this is precisely an internal functor $\mathbb{C} \to \mathbb{D}$. □

Lemma 3.4.7. *Let $\mathbb{C}$, $\mathbb{D}$ be categories and suppose that $\mathbb{C}$ is path connected. Then the map of sets*

$$Cat(\mathcal{E})(\mathbb{C}, \mathbb{D}) \xrightarrow{p} PreCat(\mathcal{E})(p(\mathbb{C}), p(\mathbb{D}))$$

induced by the application of the functor p is an injection.

Proof. Suppose that $\psi, \psi' : \mathbb{C} \to \mathbb{D}$ are two internal functors such that $p(\psi) = p(\psi')$. We need to show that $\psi = \psi'$ and for this it will suffice to show that the component $l_{\mathbb{C}}$ of the natural transformation l is an epimorphism. Indeed suppose that $l_{\mathbb{C}}$ is an epimorphism. Since the diagram

$$\begin{array}{ccc} p(\mathbb{C}) & \xrightarrow{p(\psi)=p(\psi')} & p(\mathbb{D}) \\ \downarrow{\scriptstyle l_{\mathbb{C}}} & & \downarrow{\scriptstyle l_{\mathbb{D}}} \\ y(\mathbb{C}) & \overset{y(\psi)}{\underset{y(\psi')}{\rightrightarrows}} & y(\mathbb{D}) \end{array}$$

is serially commutative we obtain that $y(\psi) = y(\psi')$. But by Lemma 3.4.6 this implies that $\psi = \psi'$ as required.

So it remains to check that $l_{\mathbb{C}}$ is an epimorphism. Since $l_{\mathbb{C}}$ is a diagram homomorphism it will suffice to show that the components of $l_{\mathbb{C}}$ on the generators of Pre are epimorphisms. The component on 'objects' is $1_{\mathbb{C}^1}$. The component on 'arrows' is $\mathbb{C}^l : \mathbb{C}^{\mathbb{I}} \to \mathbb{C}^{\mathbf{2}}$ which is an epimorphism because $\mathbb{C}$ is path connected. The component on 'equivalences' is $\mathbb{C}^{\iota\iota_1 l} : \mathbb{C}^{\mathbb{O}} \to \mathbb{C}^{\mathbf{2}}$ which is an epimorphism because $\mathbb{C}^{\iota\iota_1}$ is an epimorphism split by the map taking an element $\phi \in \mathbb{C}^{\mathbb{I}}$ to the constant element of $\mathbb{C}^{\mathbb{O}}$ at ϕ. Similarly the components $\mathbb{C}^{2*l} : \mathbb{C}^{2*\mathbb{I}} \to \mathbb{C}^{2*\mathbf{2}}$ and $\mathbb{C}^{2*\iota\iota_1 l} : \mathbb{C}^{2*\mathbb{I}} \to \mathbb{C}^{2*\mathbf{2}}$ are epimorphisms and $l_{\mathbb{C}}$ is an epimorphism as required. $\square$

Definition 3.4.8. Let $\mathbb{C}$ be a category in $\mathcal{E}$. Then the path category $P(\mathbb{C})$ of $\mathbb{C}$ is defined to be the quotient category of the path precategory of $\mathbb{C}$. That is to say we define the functor P to be equal to $\overline{(-)} \circ p$. It is clear that $\overline{(-)} \circ y \cong 1_{Cat(\mathcal{E})}$ and so the natural transformation $l : p \Rightarrow y$ induces a natural transformation $L = \overline{(-)} \circ l : P \Rightarrow 1_{Cat(\mathcal{E})}$.

Lemma 3.4.9. *Let $\mathbb{C}$ be a path connected groupoid. Then the component $L_{\mathbb{C}}$ is an epimorphism.*

Proof. The internal functor $L_{\mathbb{C}}$ is an isomorphism on objects so it will suffice to prove that it is an epimorphism on arrows. Recall that the arrow part $L^{\mathbf{2}}_{\mathbb{C}}$ is defined by the factorisation:

$$\begin{array}{ccccc} \mathbb{C}^{\mathbb{O}} & \overset{\mathbb{C}^{\iota\iota_1}}{\underset{\mathbb{C}^{\iota\iota_2}}{\rightrightarrows}} & \mathbb{C}^{\mathbb{I}} & \xrightarrow{q} & \overline{\mathbb{C}} \\ {\scriptstyle \mathbb{C}^{l\iota\iota_1}}\downarrow & & \downarrow{\scriptstyle \mathbb{C}^{l}} & & \downarrow{\scriptstyle L^{\mathbf{2}}_{\mathbb{C}}} \\ \mathbb{C}^{\mathbf{2}} & \overset{1_{\mathbb{C}^{\mathbf{2}}}}{\underset{1_{\mathbb{C}^{\mathbf{2}}}}{\rightrightarrows}} & \mathbb{C}^{\mathbf{2}} & \xrightarrow{1_{\mathbb{C}^{\mathbf{2}}}} & \mathbb{C}^{\mathbf{2}} \end{array}$$

But by the hypothesis that $\mathbb{C}$ is path connected the arrow $\mathbb{C}^l$ is an epimorphism; hence $L^{\mathbf{2}}_{\mathbb{C}}$ is an epimorphism. □

Proposition 3.4.10. *For all simply connected categories $\mathbb{C}$ in $\mathcal{E}$ the following diagram*

$$\mathbb{C}^{\mathbb{O}} \underset{\mathbb{C}^{\iota\iota_2}}{\overset{\mathbb{C}^{\iota\iota_1}}{\rightrightarrows}} \mathbb{C}^{\mathbb{I}} \xrightarrow{\mathbb{C}^l} \mathbb{C}^{\mathbf{2}}$$

is a coequaliser in $\mathcal{E}$.

Proof. Since $\mathcal{E}$ is a topos we have that the epimorphism $\mathbb{C}^l$ is the coequaliser of its kernel pair (k_1, k_2). Now k_1, k_2 are defined by the following pullback:

$$\begin{array}{ccc} Q & \xrightarrow{k_1} & \mathbb{C}^{\mathbb{I}} \\ \downarrow{\scriptstyle k_2} & & \downarrow{\scriptstyle \mathbb{C}^l} \\ \mathbb{C}^{\mathbb{I}} & \xrightarrow{\mathbb{C}^l} & \mathbb{C}^{\mathbf{2}} \end{array}$$

But because O is by definition the pushout of l along itself we know that $Q \cong \mathbb{C}^O$ and $k_i \cong \mathbb{C}^{\iota_i}$ for $i = 1, 2$. Hence

$$\mathbb{C}^{O} \underset{\mathbb{C}^{\iota_2}}{\overset{\mathbb{C}^{\iota_1}}{\rightrightarrows}} \mathbb{C}^{\mathbb{I}} \xrightarrow{\mathbb{C}^l} \mathbb{C}^{\mathbf{2}}$$

is a coequaliser. Then the result follows immediately from the fact that $\mathbb{C}^{\mathbb{O}} \xrightarrow{\mathbb{C}^{\iota}} \mathbb{C}^O$ is an epimorphism which is the hypothesis that $\mathbb{C}$ is simply connected. □

Corollary 3.4.11. *By Proposition 3.4.10 if $\mathbb{C}$ is simply connected then*

$$\mathbb{C}^{\mathbb{O}} \underset{\mathbb{C}^{\iota\iota_2}}{\overset{\mathbb{C}^{\iota\iota_1}}{\rightrightarrows}} \mathbb{C}^{\mathbb{I}} \xrightarrow{\mathbb{C}^l} \mathbb{C}^{\mathbf{2}}$$

is a coequaliser. Hence if $\mathbb{C}$ is simply connected then $L_{\mathbb{C}}$ is an isomorphism.

Remark 3.4.12. We can make groupoid versions of the path category and precategory constructions above with little extra effort. We can replace $Cat(\mathcal{E})$ with $Gpd(\mathcal{E})$, $\mathbb{I}$ with ∇I, $\mathbb{O}$ with ∇B and the definitions of path and simply connectedness with their appropriate counterparts for groupoids; the arguments of this section then go through unchanged. We first describe the additional structure that is required to define the representing object for path pregroupoids.

Definition 3.4.13. The functor $\mathcal{I}' : Pre' \to Gpd(\mathcal{E})^{op}$ is defined as the groupoid version of $\mathcal{I}$ on the subcategory $Pre \rightarrowtail Pre'$ and on the extra objects as follows. The arrows $\mathcal{I}(i_C)$ and $\mathcal{I}(i_R)$ are induced from the arrows

$$x \mapsto 1 - x : I \to I$$

and

$$(x, y) \mapsto (1 - x, y) : I \times I \to I \times I$$

respectively. Since $2_*\nabla I \cong \nabla(2_*I)$ is simply connected we have unique arrows ψ_1 and ψ_2 making the diagrams

$$\begin{array}{ccc} & & \nabla I \\ & \overset{\psi_2}{\nearrow} & 1_{\nabla I}\uparrow\uparrow\mu(\mathcal{I}(i_C),1) \\ \nabla B & \underset{\iota_2}{\overset{\iota_1}{\leftleftarrows}} & \nabla I \end{array} \quad \text{and} \quad \begin{array}{ccc} & & \nabla I \\ & \overset{\psi_1}{\nearrow} & 1_{\nabla I}\uparrow\uparrow\mu(1,\mathcal{I}(i_C)) \\ \nabla B & \underset{\iota_2}{\overset{\iota_1}{\leftleftarrows}} & \nabla I \end{array}$$

commute. Then we set $\mathcal{I}(I_1) = \psi_1$ and $\mathcal{I}(I_2) = \psi_2$.

Extending the analogy we can obtain functors corresponding to p, P and natural transformations l and L and the following results.

Proposition 3.4.14. *If $\mathbb{G}$ is a path connected groupoid then $L_{\mathbb{G}}$ is an epimorphism; if $\mathbb{G}$ is simply connected then $L_{\mathbb{G}}$ is an isomorphism.*

Proposition 3.4.15. *Let $\mathbb{G}$, $\mathbb{H}$ be groupoids and suppose that $\mathbb{G}$ is path connected. Then the map of sets*

$$Gpd(\mathcal{E})(\mathbb{G}, \mathbb{H}) \xrightarrow{p} PreGpd(\mathcal{E})(p(\mathbb{G}), p(\mathbb{H}))$$

induced by the application of the functor p is an injection.

3.5 The Weinstein Category and Precategory

The Weinstein category and precategory constructions are very similar to the constructions of the path category and precategory we have just seen but with $\mathbb{I}_\infty$ and $\mathbb{O}_\infty$ replacing the categories $\mathbb{I}$ and $\mathbb{O}$ respectively. However to define the associativity and unit arrows required in the definition of precategory we first need to check that when we take the jet part of the category $n_*\mathbb{I}$ we get the category $n_*\mathbb{I}_\infty$.

Lemma 3.5.1. *There is an inclusion of preorders*

$$\begin{array}{ccc} \mathbb{N}_I^{\mathbf{2}} & \rightarrowtail & \mathbb{I}^{\mathbf{2}} \\ & \searrow \quad \swarrow & \\ & I \times I & \end{array}$$

Proof. Let $(a,b) \in \mathbb{N}_I^{\mathbf{2}}$ such that $a \sim b$. This means that there exists a $D \in Spec(Weil)$, a $\phi \in N_I^D$ and a $d \in D$ such that $\phi(0) = a$ and $\phi(d) = b$. Hence the object D, the element $(d \mapsto (a, \phi(d))) \in (N_I \times N_I)^D$ and the element $d \in D$ gives us that $(a,a) \sim (a,b)$ in $I \times I$. Iterating this argument we obtain that if $a \approx b$ then $(a,a) \approx (a,b)$. But since $\mathbb{I}$ is a category we have that $(a,a) \in \mathbb{I}^{\mathbf{2}}$ and so since $\mathbb{I}^{\mathbf{2}}$ is jet closed in $I \times I$ by Remark 2.3.6 we see that $(a,b) \in \mathbb{I}^{\mathbf{2}}$ also. □

Corollary 3.5.2. *Combining Lemmas 2.3.18 and 3.5.1 we see that*

$$\mathbb{I}_\infty = \mathbb{N}_I$$

Using the obvious inclusion $\mathbb{N}_I^{\mathbf{2}} \rightarrowtail I^2$ *we see that*

$$\nabla I_\infty = \mathbb{N}_I$$

also and hence that $\mathbb{I}$ *and* ∇I *have equal jet parts.*

Lemma 3.5.3. *We have an inclusion of preorders*

$$\begin{array}{ccc} \mathbb{N}_B^{\mathbf{2}} & \rightarrowtail & \mathbb{O}^{\mathbf{2}} \\ & \searrow \quad \swarrow & \\ & B \times B & \end{array}$$

Proof. First we note that if $q : I \times I \twoheadrightarrow B$ is the quotient defining B in Definition 3.3.7 then $q^{D_W} : I^{D_W} \times I^{D_W} \twoheadrightarrow B^{D_W}$ is an epimorphism for all $D_W \in Spec(Weil)$ using Proposition 1.3.8. Hence if $a, b \in B$ and the object $D_W \in Spec(Weil)$, the element $\phi \in B^{D_W}$ and the element $d \in D_W$ witness that $a \sim b$ then there exists $\psi \in (I \times I)^{D_W}$ and $a', b' \in I^2$ such that $qa' = a$, $qb' = b$ and the triple (D_W, ψ, d) witnesses that $a' \sim b'$. Now the result follows from the following implications:-

$$\begin{aligned} a \sim_B b &\iff \exists (a', b') \in (I \times I)^2.\ (a' \sim b') \wedge (qa' = a) \wedge (qb' = b) \\ &\implies \exists (a', b') \in (\mathbb{I} \times \mathbb{I})^{\mathbf{2}}.\ (qa' = a) \wedge (qb' = b) \\ &\implies (a,b) \in \mathbb{O}^{\mathbf{2}} \end{aligned}$$

where the second line is obtained from the first using Lemma 3.5.1. □

Notation 3.5.4. Let RR denote the category of reflexive relations internal to $\mathcal{E}$. We write (R, X) as a shorthand for a reflexive relation $R \rightarrowtail X \times X$.

Definition 3.5.5. Recall that a category $\mathbb{C}$ in $\mathcal{E}$ is a preorder iff the arrow $(s, t) : C \to M \times M$ is a monomorphism. Let $\iota : PO \hookrightarrow Cat(\mathcal{E})$ be the full subcategory on the objects that are preorders.

Lemma 3.5.6. *The category PO of preorders is a reflexive subcategory of the category RR of reflexive relations.*

$$RR \underset{U}{\overset{tr}{\underset{\hookleftarrow}{\overset{\rightarrow}{\perp}}}} PO$$

Proof. The transitive closure $tr(R, X)$ of a reflexive relation (R, X) is the category

$$\overline{tr}R \times_X \overline{tr}R \xrightarrow{(\pi_1,\pi_4)} \overline{tr}R \overset{\xrightarrow{\pi_1}}{\underset{\xrightarrow[\pi_2]{}}{\xleftarrow{\Delta}}} X$$

where the object $\overline{tr}R$ is the internal transitive closure of R in $\mathcal{E}$. The functor U takes a groupoid that is a preorder and returns the underlying reflexive relation.

Let (R, X) be a reflexive relation. We choose the unit of the adjunction to be the inclusion $c_R = (R, X) \rightarrowtail (tr(R), X)$ of a reflexive relation into its transitive closure. Let $\mathbb{X}$ be a preorder. Then $tr(U(\mathbb{X})) = \mathbb{X}$ because all preorders are closed under composition and so we choose the counit to be $1_{\mathbb{X}}$. We check that $U(1_{\mathbb{X}}) \circ c_{U(\mathbb{X})} = 1_{U(\mathbb{X})}$ and $1_{tr(R)} \circ tr(c_R) = 1_{tr(R)}$. □

Remark 3.5.7. We now recall how to calculate a pushout

$$\begin{array}{ccc} (R_3, X_3) & \longrightarrow & (R_1, X_1) \\ \downarrow & & \downarrow \iota_1 \\ (R_2, X_2) & \xrightarrow{\iota_2} & (R, X) \end{array}$$

in RR. The object of objects is given by the pushout $X = X_2 +_{X_1} X_3$ in $\mathcal{E}$ and the subobject R consists of all the pairs $(x, y) \in X^2$ such that

$$(x, y) \in R \iff \bigvee_{i=1}^{2} (\exists x_i, y_i \in R_i.\ (\iota_i x_i = x) \wedge (\iota_i y_i = y))$$

holds in the internal logic of $\mathcal{E}$.

Lemma 3.5.8. *There is an isomorphism* $\mathbb{N}_A \cong \mathbb{N}_{A_1} +_{\mathbb{N}_{A_3}} \mathbb{N}_{A_2}$ *in PO for all pushouts*

$$\begin{array}{ccc} A_3 & \longrightarrow & A_1 \\ \downarrow & & \downarrow{\scriptstyle \iota_1} \\ A_2 & \xrightarrow{\iota_2} & A \end{array}$$

in $\mathcal{E}$.

Proof. It will suffice to show that the square

$$\begin{array}{ccc} (\sim_{A_3}, A_3) & \longrightarrow & (\sim_{A_1}, A_1) \\ \downarrow & & \downarrow{\scriptstyle \iota_1} \\ (\sim_{A_2}, A_2) & \xrightarrow{\iota_2} & (\sim_A, A) \end{array}$$

is a pushout in RR because then we could apply the functor tr to this square and use Lemma 3.5.6 to get the result. Since both of the objects $\sim_{A_1}$ and $\sim_{A_2}$ are subobjects of A^2 it will in turn suffice to show that the proposition

$$x \sim_A y \iff \vee_{i=1}^2 \exists a_i, b_i \in A_i.\ (a_i \sim b_i) \wedge (\iota_i a_i = x) \wedge (\iota_i b_i = y)$$

holds in the internal logic of $\mathcal{E}$. Now for all $D_W \in Spec(Weil)$ we have the isomorphism $A^{D_W} \cong A_1^{D_W} +_{A_3^{D_W}} A_2^{D_W}$ by Proposition 1.3.8 and so

$$\phi \in A^{D_W} \iff \vee_{i=1}^2 \exists \psi_i \in A_i^{D_W}.\ \iota_i \psi_i = \phi$$

and hence

$$\begin{aligned} x \sim_A y &\iff \bigvee_W \exists \phi \in A^{D_W}.\ \exists d \in D_W.\ (\phi(0) = x) \wedge (\phi(d) = y) \\ &\iff \bigvee_W \exists d \in D_W.\ \vee_{i=1}^2 \left(\exists \psi_i \in A_i^{D_W}.(\iota_i \psi_i(0) = x) \wedge (\iota_i \psi_i(d) = y)\right) \\ &\iff \vee_{i=1}^2 \exists a_i, b_i \in A_i.\ (a_i \sim_{A_i} b_i) \wedge (\iota_i a_i = x) \wedge (\iota_i b_i = y) \end{aligned}$$

as required. □

Corollary 3.5.9. *We have an isomorphism*

$$2_* \mathbb{I}_\infty \cong (2_* \mathbb{I})_\infty$$

Proof. We have the sequence of isomorphisms:

$$\begin{aligned}(2_*\mathbb{I})^{\mathbf{2}}_\infty &= \mathbb{N}^{\mathbf{2}}_{2_*I} \cap (2_*\mathbb{I})^{\mathbf{2}} && \text{by Lemmas 2.3.18 and 3.2.4}\\ &\cong (2_*\mathbb{N}_I)^{\mathbf{2}} \cap (2_*\mathbb{I})^{\mathbf{2}} && \text{by Corollary 3.5.8}\\ &\cong (2_*\mathbb{N}_I)^{\mathbf{2}} && \text{by Lemma 3.5.1}\\ &\cong 2_*(\mathbb{I}_\infty)^{\mathbf{2}} && \text{by Lemmas 3.2.5 and 3.5.6}\end{aligned}$$

□

Corollary 3.5.10. *We have an isomorphism* $2_*\mathbb{I}_\infty \cong (2_*\nabla I)_\infty$.

Proof. We proceed in the same way as in Corollary 3.5.9 except that rather than referring to Lemma 3.5.1 we use the inclusion $\mathbb{N}_X \rightarrowtail X \times X$. □

Corollary 3.5.11. *We have an isomorphism of categories* $(2_*\mathbb{O})_\infty \cong 2_*\mathbb{O}_\infty$.

Proof. The result follows from the sequence of isomorphisms

$$\begin{aligned}(2_*\mathbb{O})_\infty &\cong (\mathbb{N}_{2_*B})^{\mathbf{2}} \cap (2_*\mathbb{O})^{\mathbf{2}}\\ &\cong (2_*\mathbb{N}_B)^{\mathbf{2}} \cap (2_*\mathbb{O})^{\mathbf{2}}\\ &\cong (2_*\mathbb{N}_B)^{\mathbf{2}}\\ &\cong 2_*(\mathbb{N}_B^{\mathbf{2}} \cap \mathbb{O}^{\mathbf{2}})\\ &\cong 2_*\mathbb{O}_\infty\end{aligned}$$

where the first isomorphism is Lemma 2.3.18, the second is from Lemma 3.5.8 and Lemma 3.2.5, the third and fourth from Lemma 3.5.3 and the last from Corollary 2.3.18 again. □

Now that we have checked that $(-)_\infty$ preserves the appropriate pushouts we are in a position to define the 'representing diagram' for Weinstein precategories.

Definition 3.5.12. The functor $\mathcal{I}_\infty : Pre \to Cat(\mathcal{E})^{op}$ takes

$$R \underset{r_2}{\overset{r_1}{\rightrightarrows}} \xleftarrow{c} C \underset{t}{\overset{s}{\rightrightarrows}} \xleftarrow{e} M \;\mapsto\; \mathbb{O}_\infty \underset{\iota_2}{\overset{\iota_1}{\leftleftarrows}} \xrightarrow{\pi} \mathbb{I}_\infty \underset{1}{\overset{0}{\leftleftarrows}} \xrightarrow{!} 1,$$

with $\mathcal{I}_\infty(m) = \mu_\infty : \mathbb{I}_\infty \to 2_*\mathbb{I}_\infty$ and $\mathcal{I}_\infty(n) = (\mu_2)_\infty : \mathbb{O}_\infty \to 2_*\mathbb{O}_\infty$. Let $\mathbb{I}(a)$, $\mathcal{I}(\lambda)$ and $\mathcal{I}(\rho)$ be the arrows obtained in Definition 3.4.2. Then we define $\mathcal{I}_\infty(a) = \mathcal{I}(a)_\infty$, $\mathcal{I}_\infty(\lambda) = \mathcal{I}(\lambda)_\infty$ and $\mathcal{I}_\infty(\rho) = \mathcal{I}(\rho)_\infty$. Note that these arrows have the correct codomain by Corollaries 3.5.9 and 3.5.11.

Definition 3.5.13. The natural transformation $v' : \mathcal{I}_\infty \Rightarrow \mathcal{I}$ is induced from the diagram homomorphism

$$\begin{array}{ccccc}
\mathbb{O}_\infty & \begin{array}{c} \xleftarrow{\iota_1} \\ \xrightarrow{\pi} \\ \xleftarrow[\iota_2]{} \end{array} & \mathbb{I}_\infty & \begin{array}{c} \xleftarrow{0} \\ \xrightarrow{!} \\ \xleftarrow[1]{} \end{array} & 1 \\
\downarrow{\scriptstyle \iota_O} & & \downarrow{\scriptstyle \iota_I} & & \downarrow{\scriptstyle 1_1} \\
\mathbb{O} & \begin{array}{c} \xleftarrow{\iota_1} \\ \xrightarrow{\pi} \\ \xleftarrow[\iota_2]{} \end{array} & \mathbb{I} & \begin{array}{c} \xleftarrow{0} \\ \xrightarrow{!} \\ \xleftarrow[1]{} \end{array} & 1.
\end{array}$$

Remark 3.5.14. Using the $\mathcal{E}$-valued hom in $Cat(\mathcal{E})$ we have a functor

$$[Pre, Cat(\mathcal{E})^{op}]_{lex} \times Cat(\mathcal{E}) \to [Pre, \mathcal{E}]_{lex}$$

Therefore $\mathcal{I}_\infty$ induces a functor $w : Cat \to PreCat(\mathcal{E})$ respectively and v' induces a natural transformation $v : p \Rightarrow w$. If $\mathbb{C}$ is a category in $\mathcal{E}$ then $w(\mathbb{C})$ is called the Weinstein precategory of $\mathbb{C}$.

Definition 3.5.15. Let $\mathbb{C}$ be a category in $\mathcal{E}$. Then the Weinstein category $W(\mathbb{C})$ of $\mathbb{C}$ is defined to be the quotient category of the Weinstein precategory of $\mathbb{C}$. That is to say we define the functor W to be equal to $\overline{(-)} \circ w$. The natural transformation $v : p \Rightarrow w$ now induces a natural transformation $V = \overline{(-)} \circ v : P \Rightarrow W$.

Remark 3.5.16. We can make groupoid versions of the path category and precategory constructions above with little extra effort. We can replace Cat with Gpd, $\mathbb{I}$ with ∇I, $\mathbb{O}$ with ∇B and the definitions of path and simply connectedness with their appropriate counterparts for groupoids the arguments of this section go through unchanged. We first describe the additional structure that is required to define the representing object for Weinstein pregroupoids.

Definition 3.5.17. The functor $\mathcal{I}'_\infty : Pre' \to Gpd(\mathcal{E})^{op}$ is defined as the groupoid version of $\mathcal{I}_\infty$ on the subcategory $Pre \rightarrowtail Pre'$ and on the extra objects as follows. Let $\mathcal{I}(i_C)$, $\mathcal{I}(i_R)$, $\mathcal{I}(I_1)$ and $\mathcal{I}(I_2)$ be the arrows obtained in Definition 3.4.13. Then we define $\mathcal{I}_\infty(i_C) = \mathcal{I}(i_C)_\infty$, $\mathcal{I}_\infty(i_R) = \mathcal{I}(i_R)_\infty$, $\mathcal{I}_\infty(I_1) = \mathcal{I}(I_1)_\infty$ and $\mathcal{I}_\infty(I_2) = \mathcal{I}(I_2)_\infty$.

Extending the analogy further we can obtain functors corresponding to w, W and natural transformations v and V.

Chapter 4

Synthetic Lie Theory

Recall that in classical Lie theory we have a functor

$$LieGp \xrightarrow{CBH} FGL$$

taking a Lie group G to the formal group law produced by applying the Campbell-Baker-Hausdorff formula to its Lie algebra $\mathfrak{g}$. If we restrict its domain to the category of simply connected Lie groups the functor CBH becomes full, faithful and essentially surjective. In Sections 2.3.2 and 2.3.3 we defined the categories Cat_∞ and Cat_{int} that will replace the category of formal group laws and the category of Lie groups respectively in our treatment of Lie theory. In addition we constructed an adjunction

$$Cat_\infty \underset{(-)_\infty}{\overset{(-)_{int}}{\rightleftarrows}} Cat_{int} \quad (\bot)$$

and in Section 3.3 we chose a natural definition of simply connected category. In the first section of this chapter we will prove Lie's second theorem in this context: that the functor $(-)_\infty$ is full and faithful when we restrict to simply connected categories. In addition we show how (by adding an extra condition that we identified in Section 2.3.2) we can specialise this result to the situation where we replace categories in $\mathcal{E}$ with groupoids in $\mathcal{E}$.

In the second section of this chapter we describe a category Cat_∞^{sym} which we can use as an alternative to Cat_∞ provided that we work in the Cahiers topos. The objects of this category will be called symmetric jet categories.

Most of the theory for symmetric jet categories works in a completely analogous way to the theory involving asymmetric jet categories but there are considerable simplifications when defining the analogue of the functor $(-)_\infty$. These simplifications are due to the fact that jet dense arrows in the Cahiers topos are stable under pullback (which we proved in Section 2.2.4). In the final section we will show that neither the symmetric or asymmetric version of the functor $(-)_\infty$ is essentially surjective and speculate on how this might be remedied.

4.1 Lie's Second Theorem

Now we will prove the main result of the thesis: that under certain conditions on $\mathbb{C}$ and $\mathbb{X}$ any internal functor $\phi : \mathbb{C}_\infty \to \mathbb{X}$ can be lifted to an (unique) internal functor $\psi : \mathbb{C} \to \mathbb{D}$ such that

$$\begin{array}{ccc} \mathbb{C}_\infty & \xrightarrow{\phi} & \mathbb{X} \\ {\scriptstyle \iota^{\mathbb{C}}_\infty}\big\downarrow & \nearrow{\scriptstyle \psi} & \\ \mathbb{C} & & \end{array}$$

commutes. We will deduce this from the more general Theorem 4.1.6 that under certain conditions on $\mathbb{C}$ finds a unique filler for squares of the form

$$\begin{array}{ccc} \mathbb{C}_\infty & \xrightarrow{\phi} & \mathbb{X} \\ {\scriptstyle \iota^{\mathbb{C}}_\infty}\big\downarrow & \nearrow{\scriptstyle \psi} & \big\downarrow{\scriptstyle r} \\ \mathbb{C} & \xrightarrow{\xi} & \mathbb{Y} \end{array} \tag{4.1}$$

whenever r is in the right class of the integral factorisation system. In other words, we show that $\mathbb{C}_\infty \rightarrowtail \mathbb{C}$ is in the left class of the integral factorisation system. However to introduce the ideas we fix $r =! : \mathbb{X} \to 1$ which means that the condition $r \in R_{int}$ reduces to the assertion that the arrow $\mathbb{X}^{\iota_\infty} : \mathbb{X}^{\mathbb{I}_\infty} \to \mathbb{X}^{\mathbb{I}}$ is an isomorphism. In particular we have that the path groupoid $P(\mathbb{X})$ of $\mathbb{X}$ and the Weinstein groupoid $W(\mathbb{X})$ of $\mathbb{X}$ are isomorphic. This means that we can construct homomorphisms into the path groupoid $P(\mathbb{X})$ using only the infinitesimal data available in the Weinstein groupoid $W(\mathbb{X})$. We can then postcompose with the map $L_{\mathbb{X}} : P(\mathbb{X}) \to \mathbb{X}$ to obtain a map into $\mathbb{X}$.

The conditions we impose on $\mathbb{C}$ can now be motivated as those required to legitimately convert the lifting problem (4.1) into a lifting problem involving path categories. We assert that $\mathbb{C}$ is simply connected so that homomorphisms out of $\mathbb{C}$ are the same thing as homomorphisms out of $P(\mathbb{C})$. We are now in a position to state that (for the special case of $r = ! : \mathbb{X} \to 1$) the composite

$$\psi = \mathbb{C} \xrightarrow{L_{\mathbb{C}}^{-1}} P(\mathbb{C}) \xrightarrow{V_{\mathbb{C}}} W(\mathbb{C}) \cong W(\mathbb{C}_\infty) \xrightarrow{W(\phi)} W(\mathbb{X}) \cong P(\mathbb{X}) \xrightarrow{L_{\mathbb{X}}} \mathbb{X} \tag{4.2}$$

is the ψ that we require. It remains to ensure that restriction along $P(\iota_\infty^{\mathbb{C}})$ induces the correct restriction along $\iota_\infty^{\mathbb{C}}$ and we will see that it always does if we assume that $\mathbb{C}_\infty$ is path connected.

4.1.1 Proof of the Main Theorem

As indicated in the introduction to this chapter we will now fix a category $\mathbb{C}$ in $\mathcal{E}$ such that $\mathbb{C}$ itself is simply connected and the jet part $\mathbb{C}_\infty$ of $\mathbb{C}$ constructed in Definition 2.3.13 is path connected. In addition we will fix an internal functor $r : \mathbb{X} \to \mathbb{Y}$ in the right class R_{int} of the integral factorisation system. Our objective is to show that the square (4.1) has a unique filler. Note that in the case $r = ! : \mathbb{X} \to 1$ this amounts to the condition that the arrow $\mathbb{X}^{\iota_\infty} : \mathbb{X}^{\mathbb{I}_\infty} \to \mathbb{X}^{\mathbb{I}}$ is invertible and we find ourselves in the special case which we considered in the introduction to this chapter. To prove the result for a general $r \in R_{int}$ it will be convenient to modify slightly the strategy outlined above. To see why recall that $r \in R_{int}$ iff

$$\begin{array}{ccc} \mathbb{X}^{\mathbb{I}} & \xrightarrow{\mathbb{X}^{\iota_I}} & \mathbb{X}^{\mathbb{I}_\infty} \\ \downarrow{\scriptstyle r^{\mathbb{I}_\infty}} & & \downarrow{\scriptstyle r^{\mathbb{I}_\infty}} \\ \mathbb{Y}^{\mathbb{I}} & \xrightarrow{\mathbb{Y}^{\iota_I}} & \mathbb{Y}^{\mathbb{I}_\infty} \end{array} \tag{4.3}$$

is a pullback in $\mathcal{E}$. The additional difficulty we face in this general situation is that it is not obvious that the square

$$\begin{array}{ccc} P(\mathbb{X}) & \xrightarrow{V_X} & W(\mathbb{X}) \\ \downarrow{\scriptstyle P(r)} & & \downarrow{\scriptstyle W(r)} \\ P(\mathbb{Y}) & \xrightarrow{V_Y} & W(\mathbb{Y}) \end{array}$$

is a pullback. (Previously it was obvious that $\mathbb{X}^{\mathbb{I}_\infty} \cong \mathbb{X}^{\mathbb{I}}$ implied that $P(\mathbb{X}) \cong W(\mathbb{X})$.) Our solution will be to take one step further back and transfer the problem into the category of precategories where pullbacks are computed componentwise; so to show that the square

$$\begin{array}{ccc} p(\mathbb{X}) & \xrightarrow{v_X} & w(\mathbb{X}) \\ \downarrow{\scriptstyle p(r)} & & \downarrow{\scriptstyle w(r)} \\ p(\mathbb{Y}) & \xrightarrow{v_Y} & w(\mathbb{Y}) \end{array}$$

is a pullback it will suffice to check that the square

$$\begin{array}{ccc} \mathbb{X}^{\mathbb{O}} & \xrightarrow{\mathbb{X}^{\iota_I}} & \mathbb{X}^{\mathbb{O}_\infty} \\ \downarrow{\scriptstyle r^{\mathbb{O}_\infty}} & & \downarrow{\scriptstyle r^{\mathbb{O}_\infty}} \\ \mathbb{Y}^{\mathbb{O}} & \xrightarrow{\mathbb{Y}^{\iota_O}} & \mathbb{Y}^{\mathbb{O}_\infty} \end{array} \tag{4.4}$$

is a pullback which follows from the following two Lemmas.

Lemma 4.1.1. *The square*

$$\begin{array}{ccc} 2 \times \mathbb{I}_\infty & \xrightarrow{(0,1)\times\mathbb{I}_\infty} & \mathbb{I}_\infty \times \mathbb{I}_\infty \\ \downarrow{\scriptstyle \pi_1} & & \downarrow \\ 2 & \longrightarrow & \mathbb{O}_\infty \end{array}$$

is a pushout in $Cat(\mathcal{E})$.

Proof. The result follows from the isomorphisms

$$\begin{aligned} \mathbb{O}_\infty^{\mathbf{2}} &\cong \mathbb{N}_B^{\mathbf{2}} \cap \mathbb{O}^{\mathbf{2}} && \text{by Lemmas 2.3.18 and 3.3.9} \\ &\cong \mathbb{N}_B^{\mathbf{2}} && \text{by Lemma 3.5.3} \\ &\cong \mathbb{N}_2^{\mathbf{2}} +_{\mathbb{N}_{2\times I}^{\mathbf{2}}} \mathbb{N}_{I\times I}^{\mathbf{2}} && \text{by Lemma 3.5.8} \\ &\cong 2 +_{2\times\mathbb{I}_\infty^{\mathbf{2}}} (\mathbb{I}_\infty^{\mathbf{2}} \times \mathbb{I}_\infty^{\mathbf{2}}) \end{aligned}$$

where the last line we use the fact that

$$\mathbb{N}_{I\times I} \cong \mathbb{N}_I \times \mathbb{N}_I \cong \mathbb{I}_\infty \times \mathbb{I}_\infty$$

by Example 2.3.3. □

Lemma 4.1.2. *The arrow $\iota_\infty^{\mathbb{O}} : \mathbb{O}_\infty \to \mathbb{O}$ is in the left class of the integral factorisation system.*

Proof. Since the integral factorisation system is a $Cat(\mathcal{E})$-factorisation system and $\iota^{\mathbb{I}}_{\infty} : \mathbb{I}_{\infty} \to \mathbb{I}$ is in the left class L_{int} we see that $\iota^{\mathbb{I}}_{\infty} \times \iota^{\mathbb{I}}_{\infty}$ is in L_{int}. But L_{int} is also closed under colimits so the pushout

$$\begin{array}{ccc} \iota^{\mathbf{2}}_{\infty} \times \iota^{\mathbb{I}}_{\infty} & \xrightarrow{F \times \iota^{\mathbb{I}}_{\infty}} & \iota^{\mathbb{I}}_{\infty} \times \iota^{\mathbb{I}}_{\infty} \\ \downarrow{\scriptstyle \pi_1} & & \downarrow \\ \iota^{\mathbf{2}}_{\infty} & \longrightarrow & \iota^{\mathbb{O}}_{\infty} \end{array}$$

is also in L_{int} as required. The arrow F is the one described by the pair of arrows $((0,1) \times \mathbb{I}_{\infty}, l \times \mathbb{I})$. $\square$

Corollary 4.1.3. *For all arrows $r : \mathbb{X} \to \mathbb{Y}$ in the right class of the integral factorisation system the square*

$$\begin{array}{ccc} p(\mathbb{X}) & \xrightarrow{\upsilon_X} & w(\mathbb{X}) \\ \downarrow{\scriptstyle p(r)} & & \downarrow{\scriptstyle w(r)} \\ p(\mathbb{Y}) & \xrightarrow{\upsilon_Y} & w(\mathbb{Y}) \end{array}$$

is a pullback in $PreCat(\mathcal{E})$.

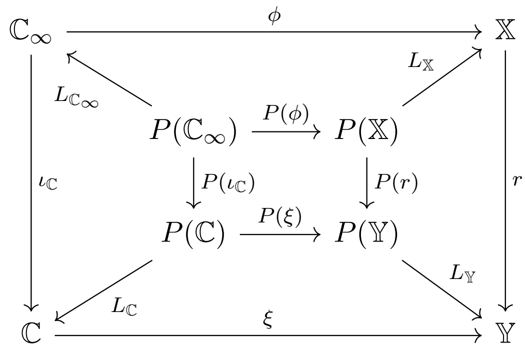
The following two Lemmas now effect the transfer of the lifting problem to the category of precategories.

Lemma 4.1.4. *Let $\mathcal{D}$ be the commutative square*

$$\begin{array}{ccc} \mathbb{C}_{\infty} & \xrightarrow{\phi} & \mathbb{X} \\ \downarrow{\scriptstyle \iota_{\mathbb{C}}} & & \downarrow{\scriptstyle r} \\ \mathbb{C} & \xrightarrow{\xi} & \mathbb{Y} \end{array}$$

in $Cat(\mathcal{E})$. If the square $P(\mathcal{D})$ has a filler then the square $\mathcal{D}$ has a filler.

Proof. Since $\mathbb{C}$ is simply connected by hypothesis then Corollary 3.4.11 implies that $L_{\mathbb{C}}$ is an isomorphism. We are also assuming that the jet part $\mathbb{C}_{\infty}$ is path connected and so using Lemma 3.4.9 we see that $L_{\mathbb{C}_{\infty}}$ is an epimorphism. By naturality of L the cube



commutes. Therefore if we have a χ that fills the inner square then $L_{\mathbb{X}} \circ \chi \circ L_{\mathbb{C}}^{-1}$ fills the outer square. Indeed it is immediate that the lower triangle commutes:

$$\begin{aligned} r \circ L_{\mathbb{X}} \circ \chi \circ L_{\mathbb{C}}^{-1} &= L_{\mathbb{Y}} \circ P(r) \circ \chi \circ L_{\mathbb{C}}^{-1} \\ &= L_{\mathbb{Y}} \circ P(\xi) \circ L_{\mathbb{C}}^{-1} \\ &= \xi \end{aligned}$$

To see that the upper triangle commutes we use that $L_{\mathbb{C}_\infty}$ is an epimorphism:

$$\begin{aligned} L_{\mathbb{X}} \circ \chi \circ L_{\mathbb{C}}^{-1} \circ \iota_{\mathbb{C}} \circ L_{\mathbb{C}_\infty} &= L_{\mathbb{X}} \circ \chi \circ P(\iota_{\mathbb{C}}) \\ &= L_{\mathbb{X}} \circ P(\phi) \\ &= \phi \circ L_{\mathbb{C}_\infty} \\ \implies L_{\mathbb{X}} \circ \chi \circ L_{\mathbb{C}}^{-1} \circ \iota_{\mathbb{C}} &= \phi \end{aligned}$$

□

Corollary 4.1.5. *If the square $p(\mathcal{D})$ has a filler then the square $\mathcal{D}$ has a filler.*

Proof. By Lemma 4.1.4 it will suffice to obtain a filler of the square $P(\mathcal{D})$. To get a filler of $P(\mathcal{D})$ from a filler of $p(\mathcal{D})$ we apply the functor $\overline{(-)} : PreCat(\mathcal{E}) \to Cat(\mathcal{E})$ that forms the quotient groupoid of the pregroupoid. □

Now the main result follows in a formal fashion. We obtain the existence of the required lifts in an analogous way to (4.2) in the introduction and prove uniqueness using the fact that p is faithful for a path connected domain.

Theorem 4.1.6. *Let $\mathbb{C}$ be a simply connected category such that $\mathbb{C}_\infty$ is path connected. Then the inclusion $\iota_{\mathbb{C}} : \mathbb{C}_\infty \rightarrowtail \mathbb{C}$ is in the left class of the integral factorisation system.*

Proof. Suppose that

$$\begin{array}{ccc} \mathbb{C}_\infty & \xrightarrow{\phi} & \mathbb{X} \\ \downarrow{\scriptstyle \iota_{\mathbb{C}}} & & \downarrow{\scriptstyle r} \\ \mathbb{C} & \xrightarrow{\xi} & \mathbb{Y} \end{array} \tag{4.5}$$

commutes. Then we need to find a unique filler $\psi : \mathbb{C} \to \mathbb{X}$. To exhibit the existence of a filler we use Corollary 4.1.5 to see that it will suffice to find a filler Ψ for the square

$$\begin{array}{ccc} p(\mathbb{C}_\infty) & \xrightarrow{p(\phi)} & p(\mathbb{X}) \\ \downarrow{\scriptstyle p(\iota_{\mathbb{C}})} & & \downarrow{\scriptstyle p(r)} \\ p(\mathbb{C}) & \xrightarrow{p(\xi)} & p(\mathbb{Y}) \end{array} \tag{4.6}$$

in $PreCat(\mathcal{E})$. To do this we will exhibit $p(\mathbb{X})$ as a pullback. By combining the hypothesis that r is in the right class of the integral factorisation system with Corollary 4.1.3 we see that the bottom right square in

$$\begin{array}{ccccc} p(\mathbb{C}) & \xrightarrow{v_{\mathbb{C}}} & w(\mathbb{C}) & \xrightarrow{w(\iota_{\mathbb{C}})^{-1}} & w(\mathbb{C}_\infty) \\ & \searrow{\scriptstyle \Psi} & & & \downarrow{\scriptstyle w(\phi)} \\ & & p(\mathbb{X}) & \xrightarrow{v_{\mathbb{X}}} & w(\mathbb{X}) \\ & {\scriptstyle p(\xi)}\searrow & \downarrow{\scriptstyle p(r)} & & \downarrow{\scriptstyle w(r)} \\ & & p(\mathbb{Y}) & \xrightarrow{v_{\mathbb{Y}}} & w(\mathbb{Y}) \end{array}$$

is a pullback. By Lemma 2.3.17 the arrow $w(\iota_{\mathbb{C}}) : w(\mathbb{C}_\infty) \to w(\mathbb{C})$ is invertible. Thus if we want to find an arrow extending $p(\phi)$ we have a natural choice: the arrow $\Psi : p(\mathbb{C}) \to p(\mathbb{X})$ induced by the pair $(p(\xi), w(\phi) \circ w(\iota_{\mathbb{C}})^{-1} \circ v_{\mathbb{C}})$ as displayed above. First we check that this pair does indeed give rise to a commuting square:

$$\begin{aligned} w(r) \circ w(\phi) \circ w(\iota_{\mathbb{C}})^{-1} \circ v_{\mathbb{C}} &= w(\xi) \circ w(\iota_{\mathbb{C}}) \circ w(\iota_{\mathbb{C}})^{-1} \circ v_{\mathbb{C}} \\ &= w(\xi) \circ v_{\mathbb{C}} \\ &= v_{\mathbb{Y}} \circ p(\xi) \end{aligned}$$

and hence gives rise to Ψ as indicated. We now check that Ψ is indeed a filler for Diagram 4.6:

1. the equality $p(r) \circ \Psi = p(\xi)$ follows immediately from the definition of Ψ,

2. the equality $\Psi \circ p(\iota_{\mathbb{C}}) = p(\phi)$ is checked using the universal property of the pullback $p(\mathbb{X})$:

   (a) firstly:

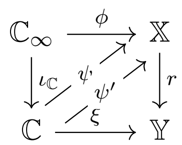
$$\begin{aligned} v_{\mathbb{X}} \circ \Psi \circ p(\iota_{\mathbb{C}}) &= w(\phi) \circ w(\iota_{\mathbb{C}})^{-1} \circ v_{\mathbb{C}} \circ p(\iota_{\mathbb{C}}) \\ &= w(\phi) \circ w(\iota_{\mathbb{C}})^{-1} \circ w(\iota_{\mathbb{C}}) \circ v_{\mathbb{C}_\infty} \\ &= w(\phi) \circ v_{\mathbb{C}_\infty} \\ &= v_{\mathbb{X}} \circ p(\phi) \end{aligned}$$

   (b) secondly:

$$p(r) \circ \Psi \circ p(\iota_{\mathbb{C}}) = p(\xi) \circ p(\iota_{\mathbb{C}}) = p(r) \circ p(\phi).$$

So by Corollary 4.1.5 we obtain a filler for (4.5).

To show that this filler is unique suppose that we have two arrows ψ and ψ' such that



(4.7)

is serially commutative. Then we need to show that $\psi = \psi'$. By hypothesis $\mathbb{C}$ is path connected so by an application of Lemma 3.4.7 it will suffice to show that $p(\psi) = p(\psi') : p(\mathbb{C}) \to p(\mathbb{X})$ and to exhibit this equality we again use the fact that $p(\mathbb{X})$ is a pullback:

1. the equality $p(r) \circ p(\psi) = p(r) \circ p(\psi')$ follows immediately from the serial commutativity of Diagram 4.7,

2. the equality $v_{\mathbb{X}} \circ p(\psi) = v_{\mathbb{X}} \circ p(\psi')$ follows from the serial commutativity

of the diagram

$$\begin{array}{ccc} p(\mathbb{C}) & \underset{p(\psi')}{\overset{p(\psi)}{\rightrightarrows}} & p(\mathbb{X}) \\ \downarrow{\scriptstyle v_{\mathbb{C}}} & & \downarrow{\scriptstyle v_{\mathbb{X}}} \\ w(\mathbb{C}) & \underset{w(\psi')}{\overset{w(\psi)}{\rightrightarrows}} & w(\mathbb{X}) \\ {\scriptstyle w(\iota_{\mathbb{C}})^{-1}}\downarrow & \nearrow{\scriptstyle w(\phi)} & \\ w(\mathbb{C}_\infty) & & \end{array}$$

where we have used Lemma 2.3.17 to see that $w(\iota_{\mathbb{C}})$ is invertible.

□

Corollary 4.1.7. *Consider the special case where $r = (s,t) : \mathbb{X} \to 1$ and so $\mathbb{X}^{\mathbb{I}_\infty} \cong \mathbb{X}^{\mathbb{I}}$. Then we have shown that any internal functor $\phi : \mathbb{C}_\infty \to \mathbb{X}$ can be lifted to a (unique) internal functor $\psi : \mathbb{C} \to \mathbb{X}$ such that*

$$\begin{array}{ccc} \mathbb{C}_\infty & \xrightarrow{\phi} & \mathbb{X} \\ {\scriptstyle \iota_C}\downarrow & \nearrow{\scriptstyle \psi} & \\ \mathbb{C} & & \end{array}$$

commutes in $Cat(\mathcal{E})$.

Definition 4.1.8. Let $Cat^{int}_{sc}(\mathcal{E})$ denote the full subcategory of $Cat(\mathcal{E})$ whose objects are simply connected categories $\mathbb{C}$ such that the jet part $\mathbb{C}_\infty$ is path connected and the arrow

$$\mathbb{C}^{\mathbb{I}} \xrightarrow{\mathbb{C}^{\iota_\infty}} \mathbb{C}^{\mathbb{I}_\infty}$$

is an isomorphism.

Corollary 4.1.9. *The functor $(-)_\infty : Cat^{int}_{sc}(\mathcal{E}) \to Cat_\infty(\mathcal{E})$ is full and faithful.*

Proof. Let $\mathbb{C}$ and $\mathbb{X}$ be simply connected categories whose jet part is path connected. To see that $(-)_\infty$ is faithful let $\phi, \psi : \mathbb{C} \to \mathbb{X}$ be internal functors and suppose that $\phi_\infty = \psi_\infty$. Then we have that $\phi\iota_C = \iota_X\phi_\infty = \iota_X\psi_\infty = \psi\iota_C$. But by Corollary 4.1.7 we see that there is a unique lift $\chi : \mathbb{C} \to \mathbb{X}$ such that $\chi\iota_C = \phi\iota_C$. Since both ϕ and ψ are examples of such lifts we have that $\phi = \chi = \psi$ as required.

To see that $(-)_\infty$ is full let $z : \mathbb{C}_\infty \to \mathbb{X}_\infty$. Then using Corollary 4.1.7 again we see that there is a (unique) homomorphism $\psi : \mathbb{C} \to \mathbb{X}$ such that $\psi\iota_C = \iota_X z$. Since ι_X is a monomorphism $\iota_X\psi_\infty = \psi\iota_C = \iota_X z$ implies $\psi_\infty = z$ as required. □

Remark 4.1.10. We have seen that it is not possible to deduce Lie's second theorem for groupoids immediately from Lie's second theorem for categories because by Corollary 2.3.22 the jet part $(\nabla D)_\infty$ of the pair groupoid on the space

$$D = \{x \in R : x^2 = 0\}$$

is a category but cannot be given the structure of a groupoid. However Propositions 2.3.20 and 2.3.26 tell us that if we impose the extra condition that the relation $\approx$ is symmetric on the space (G, s) in $\mathcal{E}/M$ then the jet part can always be made into a groupoid in a natural manner. Therefore with this restriction in place we can specialise the theory to the case of groupoids.

Definition 4.1.11. Let $Gpd^{int}_{sc,\approx}(\mathcal{E})$ denote the full subcategory of $Gpd(\mathcal{E})$ whose objects are simply connected groupoids $\mathbb{G}$ with arrow space G such that the jet part $\mathbb{G}_\infty$ is path connected, the relation $\approx$ is symmetric on the object (G, s) of $\mathcal{E}/M$ and the arrow

$$\mathbb{G}^{\nabla I} \xrightarrow{\mathbb{G}^{\iota_\infty}} \mathbb{G}^{\nabla I_\infty}$$

is an isomorphism.

Theorem 4.1.12. *The functor* $(-)_\infty : Gpd^{int}_{sc,\approx}(\mathcal{E}) \to Cat_\infty(\mathcal{E})$ *is full and faithful.*

4.1.2 The Symmetric Jet Part in the Cahiers Topos

Recall that in Section 2.3.2 we used the jet factorisation system on the slice category $\mathcal{E}/M$ when defining the jet part of a category $\mathbb{C}$ with base space M. The reason that we needed to do this was that if we defined $\mathbb{C}^{\mathbf{2}}_\infty$ as the mediating object of the jet factorisation

$$M \xrightarrow{e_\infty} C_\infty \xrightarrow{\iota_\infty} C$$

of the arrow e in $\mathcal{E}$ then the arrow

$$C_\infty \xrightarrow{(1_{C_\infty}, e_\infty t_\infty)} 2_\times C_\infty$$

is not obviously jet dense in $\mathcal{E}$. To see the difficulty let us fix an element (f, g) of $2_\times C_\infty$. Then we always have that $esg \approx g$. Heuristically speaking there is a jet ϕ containing g that has base at esg and we would like to pair this jet with f to obtain $(f, esg) \approx (f, g)$. The difficulty is that although esg and g have the same source we cannot guarantee that the whole of the jet ϕ linking them is contained in the same source fibre and hence it doesn't necessarily make sense to precompose ϕ with f. Intuitively the inability to find a source constant jet from esg to g requires an infinitesimal 'gap' or 'dent' in the arrow space of $\mathbb{C}$ so that we must change source fibre in order to go around it. Since this might be considered a pathological property for an object in a category of 'smooth spaces' to have we now show that in the Cahiers topos the arrow $(1_{C_\infty}, e_\infty t_\infty)$ actually is jet dense. Once we have done this we can define an alternative (symmetric) jet part using only the jet factorisation system on $\mathcal{E}$. Then we can proceed in a completely analogous manner with the asymmetric case to obtain a functor from simply connected integral complete categories to symmetric jet categories that is full and faithful. One convenient feature of this symmetric theory is that it specialises with little modification to the situation where we consider groupoids rather than categories.

The special property of the Cahiers topos that we will use is the pullback stability of jet dense arrows. Hence we recall Proposition 2.2.33:

Proposition. *Let $g : A \to B$ be jet dense in the Cahiers topos and*

$$\begin{array}{ccc} P & \xrightarrow{\pi_2} & A \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle g} \\ C & \xrightarrow{k} & B \end{array}$$

be a pullback in $\mathcal{E}$. Then the arrow π_1 is jet dense.

In the rest of this section we will work in the Cahiers topos $\mathcal{E}_W$ and $\mathbb{C}$ will denote an arbitrary category in $\mathcal{E}$ with arrow space C object space M and structure maps s, t, e and μ.

Corollary 4.1.13. *The arrow*

$$C_\infty \xrightarrow{(1_{C_\infty}, e_\infty t_\infty)} 2_* C_\infty$$

is jet dense in the Cahiers topos.

Proof. The arrow $e_\infty : M \to C_\infty$ is jet dense in $\mathcal{E}$ and so the pullback

$$\begin{array}{ccc} C_\infty & \xrightarrow{s_\infty} & M \\ {\scriptstyle (1_{C_\infty}, e_\infty \circ s_\infty)} \downarrow & & \downarrow {\scriptstyle e_\infty} \\ C_\infty \; {}_{s_\infty}\!\times_{t_\infty} C_\infty & \xrightarrow{\pi_2} & C_\infty \end{array}$$

shows that the arrow $(1_{C_\infty}, e_\infty \circ s_\infty) : C_\infty \to C_\infty \; {}_{s_\infty}\!\times_{t_\infty} C_\infty$ is jet dense in $\mathcal{E}$. $\square$

Definition 4.1.14. Let $\mathbb{C}$ be a category in the Cahiers topos. Then the symmetric jet part $\mathbb{C}_\infty$ of $\mathbb{C}$ is defined as follows. It has object space M, arrow space C_∞ defined as the mediating object in the jet factorisation of the arrow $e : M \to C$:

$$M \xrightarrow{e_\infty} C_\infty \xrightarrow{\iota_C} C$$

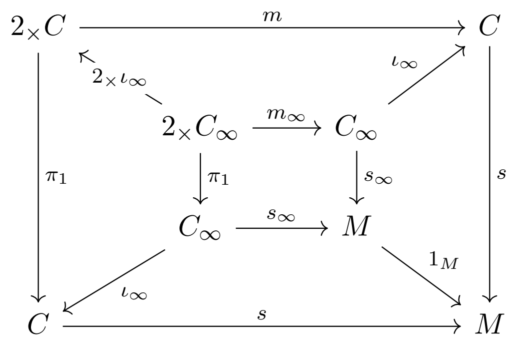
and reflexive graph structure given by

$$C_\infty \; \underset{t_\infty}{\overset{s_\infty}{\rightrightarrows}} \; M, \qquad M \xrightarrow{e_\infty} C_\infty$$

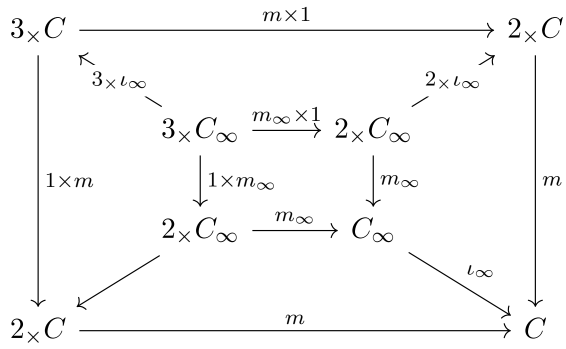
where $s_\infty = s \circ \iota_C$ and $t_\infty = t \circ \iota_C$. The multiplication $m_\infty : 2_\times C_\infty \to C_\infty$ is defined as the unique filler for the square

$$\begin{array}{ccc} C_\infty & \xrightarrow{1_{C_\infty}} & C_\infty \\ {\scriptstyle (1_{C_\infty}, e_\infty t_\infty)} \downarrow & \overset{m_\infty}{\nearrow} & \downarrow {\scriptstyle \iota_C} \\ 2_\times C_\infty & \xrightarrow[m \circ 2_\times \iota_\infty]{} & C \end{array}$$

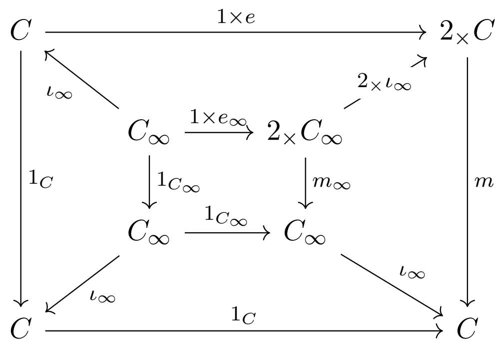
where $(1_{C_\infty}, e_\infty t_\infty)$ is jet dense by Corollary 4.1.13. The multiplication m_∞ is well-typed because the outer square of



commutes and ι_∞ is a monomorphism. The associativity axiom holds because the outer square of



commutes and ι_∞ is a monomorphism. One unit axiom holds because



commutes and ι_∞ is a monomorphism and the other unit axiom is shown to hold in a similar manner.

Remark 4.1.15. In an analogous way to the asymmetric construction in Section 2.3.2 we make the definition of a symmetric jet category and show that the function $(-)_\infty$ taking a category to its jet part extends to a functor. Then we obtain the following results.

Proposition 4.1.16. *The category* $Cat_\infty^{sym}(\mathcal{E}_W)$ *of symmetric jet categories in the Cahiers topos* $\mathcal{E}_W$ *is a coreflective subcategory of* $Cat(\mathcal{E}_W)$*.*

Theorem 4.1.17. *Let* $Cat_{sc}^{int}(\mathcal{E}_W)$ *denote the full subcategory of* $Cat(\mathcal{E}_W)$ *whose objects are simply connected categories* $\mathbb{C}$ *in the Cahiers topos such that the jet part* $\mathbb{C}_\infty$ *is path connected and the arrow*

$$\mathbb{C}^{\mathbb{I}} \xrightarrow{\mathbb{C}^{\iota_\infty}} \mathbb{C}^{\mathbb{I}_\infty}$$

is an isomorphism. Then the functor

$$Cat_{sc}^{int}(\mathcal{E}_W) \xrightarrow{(-)_\infty} Cat_\infty^{sym}(\mathcal{E}_W)$$

is full and faithful.

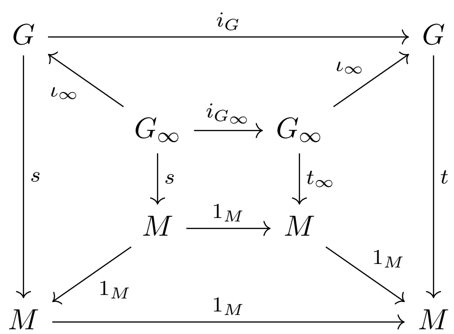
Remark 4.1.18. Unlike the construction of the asymmetric jet part, the construction of the symmetric jet part specialises from categories to groupoids without much further effort. The remainder of this section gives the appropriate commutative diagrams that make this work.

Proposition 4.1.19. *Let* $\mathbb{G}$ *be a groupoid with symmetric jet part* $\mathbb{G}_\infty$*. Then the arrow* $i_{G_\infty} : G_\infty \to G_\infty$ *defined as the unique filler for the square*

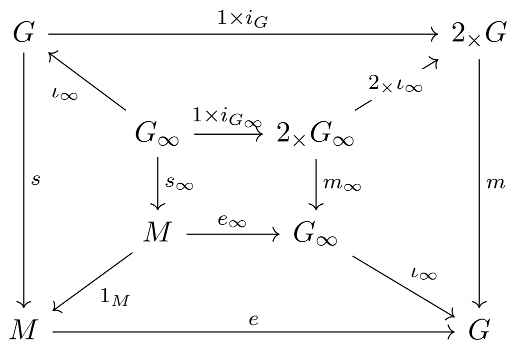
$$\begin{array}{ccc} M & \xrightarrow{e_\infty} & G_\infty \\ {\scriptstyle e_\infty}\downarrow & i_{G_\infty} & \downarrow{\scriptstyle \iota_\infty} \\ G_\infty & \xrightarrow{i_G \iota_\infty} & G \end{array}$$

is a well-defined inverse that makes $\mathbb{G}_\infty$ *into a groupoid.*

Proof. The inverse i_{G_∞} is well typed because the outer square of



commutes and ι_∞ is a monomorphism. One inverse axiom holds because



commutes and ι_∞ is a monomorphism. Similarly for the other inverse axiom.

□

Remark 4.1.20. Again we make the definition of a symmetric jet groupoid and show that the function $(-)_\infty$ taking a groupoid to its symmetric jet part extends to a functor. In this case we obtain the following results.

Proposition 4.1.21. *The category $Gpd_\infty^{sym}(\mathcal{E}_W)$ of symmetric jet groupoids in $\mathcal{E}_W$ is a coreflective subcategory of $Gpd(\mathcal{E}_W)$.*

Theorem 4.1.22. *Let $Gpd_{sc}^{int}(\mathcal{E}_W)$ denote the full subcategory of $Gpd(\mathcal{E}_W)$ whose objects are simply connected groupoids $\mathbb{G}$ such that the jet part $\mathbb{G}_\infty$ is path connected and the arrow*

$$\mathbb{G}^{\nabla I} \xrightarrow{\mathbb{G}^{\iota_\infty}} \mathbb{G}^{\nabla I_\infty}$$

is an isomorphism. Then the functor

$$Gpd_{sc}^{int}(\mathcal{E}_W) \xrightarrow{(-)_\infty} Gpd_\infty^{sym}(\mathcal{E}_W)$$

is full and faithful.

4.2 Lie's Third Theorem

The construction of the functor

$$Gpd_\infty(\mathcal{E}) \xrightarrow{(-)_{int}} Gpd_{int}(\mathcal{E})$$

in Section 2.3.3 allows us to obtain an integral complete category from any jet category. However for the corresponding situation in classical Lie theory more is true: there is an equivalence of categories between the category of formal group laws and the category of simply connected Lie groups. Now in Example 2.3.12 we have seen that jet groups of the form (D^n_∞, μ) are precisely formal group laws. Hence we can recover this equivalence of categories by restricting the domain of $(-)_{int}$ to jet groups of the form (D^n_∞, μ) and its codomain to simply connected Lie groups.

We will now describe a counterexample which shows that in the adjunction

$$Gpd_\infty(\mathcal{E}) \underset{(-)_\infty}{\overset{(-)_{int}}{\rightleftarrows}} Gpd_{int}(\mathcal{E}) \quad (\perp) \tag{4.8}$$

the unit is not an isomorphism. In other words, we will find a jet groupoid $\mathbb{K}^*$ for which $(\mathbb{K}^*_{int})_\infty$ is not isomorphic to $\mathbb{K}^*$. Recall that since the underlying vector bundle of any Lie algebroid is by definition locally trivial the dimension of its fibres are constant within connected components of the base space. The analogous property involving the vertex groups of a jet groupoid does not hold.

Example 4.2.1. The groupoid $\mathbb{K}^*$ is given by the pushout

$$\begin{array}{ccc} 1 & \xrightarrow{\;0\;} & (D_\infty, +) \\ {\scriptstyle e(1)}\downarrow & & \downarrow{\scriptstyle u_1} \\ \nabla I_\infty & \xrightarrow{\;u_2\;} & \mathbb{K}^* \end{array}$$

where $(D_\infty, +)$ is the internal group in $\mathcal{E}$ with arrow space D_∞ and composition given by addition. We first observe that the vertex group of $\mathbb{K}^*$ at the element $0 \in_1 I$ is trivial because $\nabla I_\infty = \mathbb{N}_I$ is a preorder. Next we observe that the vertex group of $\mathbb{K}^*$ at the element $1 \in_1 I$ is $(D_\infty, +)$ by construction. However when we form the integral completion $\mathbb{K}^*_{int}$ we have a lift $\psi : \nabla I \to \mathbb{K}^*_{int}$ making the diagram

$$\begin{array}{ccc} \nabla I_\infty & \xrightarrow{\;\tau u_2\;} & \mathbb{K}^*_{int} \\ {\scriptstyle \iota_\infty}\downarrow & \overset{\psi}{\nearrow} & \\ \nabla I & & \end{array}$$

commute. Then by conjugating an element of the vertex space of $\mathbb{K}^*$ at $1 \in_1 I$ with the arrow $\psi(l)$ we see that the vertex space of $\mathbb{K}^*_{int}$ at $0 \in_1 I$ is not trivial. For instance at stage of definition D we have that the arrow defined by

$$d \mapsto \psi(l)^{-1} \circ u_1(d) \circ \psi(l)$$

is a generalised element of $(\mathbb{K}^*_{int})^{\mathbf{2}}_\infty$ at stage of definition D that is an endomorphism of the object $0 \in_1 I$. Hence the groupoid $(\mathbb{K}^*_{int})_\infty$ is not isomorphic to $\mathbb{K}^*$.

Therefore the question arises: can we characterise the jet categories $\mathbb{K}$ for which the unit of the adjunction (4.8) is an isomorphism? It is a question that we will not answer in this thesis; it would be a logical direction for future research. In view of the previous example it certainly would be necessary to ensure that if there is an A-path $\phi : \nabla I_\infty \to \mathbb{K}$ starting at the object x and ending at the object y then the vertex groups at x and y are isomorphic. Therefore one would need to find some way of transporting infinitesimal arrows along A-paths. It is tempting to speculate that a condition asserting the existence of an appropriately defined notion of connection (and therefore parallel transport) for A-paths would be sufficient to obtain an equivalence of categories.

Chapter 5

Relationship to Classical Lie Theory

In the course of this thesis we have made several definitions concerning objects in a well-adapted model $\mathcal{E}$ which are intended to be analogous to certain definitions in classical Lie theory. In this chapter we will make precise how our definitions in $\mathcal{E}$ generalise their classical counterparts. For notational convenience we will use the prefix $\mathcal{E}$ when describing the non-classical constructions we have made. For example an $\mathcal{E}$-simply connected groupoid is a groupoid $\mathbb{G}$ in $\mathcal{E}$ such that the arrow

$$\mathbb{G}^{\mathbb{O}} \xrightarrow{\mathbb{G}^{\iota}} \mathbb{G}^{O}$$

is an epimorphism and an s-simply connected Lie groupoid is a Lie groupoid in Man which has simply connected source fibres.

The main difficulty is that the classical definitions only concern global elements but their synthetic counterparts must be shown to hold for all generalised elements with domain a representable object of $\mathcal{E}$. If we restrict attention to the well adapted models that we have considered in Chapter 1 then the topos $\mathcal{E}_{germ}$ has the largest number of representable objects and $\mathcal{E}_W$ the least. Although in the last section we will resort to using the simpler representable objects in $\mathcal{E}_W$ we will otherwise prove the more general results that arise from working in any well-adapted model.

In Section 5.1 we will show that all s-path connected Lie groupoids are $\mathcal{E}$-path connected in $\mathcal{E}_{germ}$. In this case the classical property of being s-path connected tells us that for any global element $g \in_1 G$ of the arrow space there is a source constant path γ that starts at esg and ends at g. By contrast to prove that $\mathbb{G}$ is $\mathcal{E}$-simply connected it turns out that we need to show that for every global element $g \in_1 G$ there exists an open set U containing g and a source constant homotopy from U to some open set completely contained in the image of the identity map e. The main idea is to use the fact that for any Lie groupoid in Man the source map s is a submersion and so by the implicit function theorem we have that s is locally isomorphic to the orthogonal projection $\mathbb{R}^{k+n} \twoheadrightarrow \mathbb{R}^k$ onto the first k coordinates for some $k, n \in \mathbb{N}$. Then intuitively speaking, if we are given an open set U we can move it closer to the image of e in a source constant manner by translating it in parallel to this orthogonal projection.

In Section 5.2 we prove that every Lie groupoid $\mathbb{G}$ the jet part $\mathbb{G}_\infty$ is path connected in $\mathcal{E}_{germ}$. To do this we find a cover of $\mathbb{G}_\infty$ in $\mathcal{E}_{germ}$ and for every subobject U in this cover a retraction of U into the image of e_∞. The cover that we will use will be a restriction of a cover of $\mathbb{G}$ on which the source map $s|_U$ is locally a projection. Although in Section 5.1 we show how to retract U onto the image of e in $\mathbb{G}$ the main work in Section 5.2 is to demonstrate that we can choose this retraction so that it factors through $\mathbb{G}_\infty$. To do this we will show that a certain arrow which arises as a pullback is jet dense and so will appeal to Corollary 2.2.3.

I have been unable to prove that every s-simply connected Lie groupoid is $\mathcal{E}$-simply connected. However in Section 5.3 we will see that the single object version of this statement is true. That is to say: every simply connected Lie group is $\mathcal{E}_{germ}$-simply connected. In Section 5.4 we describe how the definitions of A-paths and $\mathbb{G}$-paths that can be found in for example [7] are generalised in $\mathcal{E}_{germ}$ and that for every Lie groupoid $\mathbb{G}$ we have that the arrow $! : \mathbb{G} \to 1$ is in the right class of the integral factorisation system. Explicitly this means that we show that the arrow

$$\mathbb{G}^{\mathbb{I}} \xrightarrow{\mathbb{G}^{\iota_\infty}} \mathbb{G}^{\mathbb{I}_\infty}$$

is an isomorphism. To do this we reduce the problem to certain classical results concerning the solution of time- and parameter- dependent vector fields. In order to make this reduction we make use of the simplified structure of representables in the Cahiers topos to separate the infinitesimal part from the macroscopic part. Therefore we only show that $\mathbb{G}^{\mathbb{I}} \cong \mathbb{G}^{\mathbb{I}_\infty}$ in $\mathcal{E}_W$.

5.1 Path Connectedness

In this section we let $\mathcal{E}$ be any well-adapted model. We fix a Lie groupoid $\mathbb{G}$ in $\mathcal{E}$ that has arrow space G, object space M and structure maps s, t, e and μ. We need to show that for all s-path connected Lie groupoids the arrow

$$\mathbb{G}^{\nabla I} \xrightarrow{\mathbb{G}^l} \mathbb{G}^{\mathbf{2}}$$

is an epimorphism. To do this we recall Corollary 5 in III.7 of [23]:

Proposition 5.1.1. *A morphism of sheaves $\phi : F \to G$ is an epimorphism in the Grothendieck topos $Sh(\mathcal{C}, \mathcal{J})$ iff for each object C of $\mathcal{C}$ and each element $y \in G(C)$, there is a cover S of C such that for all $f : D \to C$ in S the element yf is in the image of $\phi_D : F(D) \to G(D)$.*

Therefore it will suffice to find a cover of $\mathbb{G}^{\mathbf{2}}$ such that for each U in the cover we have a lift $\psi : U \to \mathbb{G}^{\nabla I}$ such that

$$\begin{array}{ccc} & & \mathbb{G}^{\nabla I} \\ & \overset{\psi}{\nearrow} & \downarrow{\scriptstyle \mathbb{G}^l} \\ U & \xrightarrow{\iota_U} & \mathbb{G}^{\mathbf{2}} \end{array}$$

commutes. To find this cover we will perform a ‘continuous induction’ argument. First as a ‘base case’ we will find for every element $m \in M$ an open set U_{em} containing em and a smooth retraction of U_{em} into the image of e. Now consider for each $m \in M$ the set S_m consisting of all elements $g \in s^{-1}m$ such that there exists an open set U_g containing g and a source constant homotopy F from U_g into the image of e. Our ‘induction step’ will be to show that for all $m \in M$ the set S_m is both open and closed. Recall that every source fibre $s^{-1}m$ is locally connected because it is a smooth manifold. This means that $s^{-1}m$ is path connected iff $s^{-1}m$ is connected. Since S_m is non-empty by the base case for s-path connected $\mathbb{G}$ the set S_m is the whole of $s^{-1}m$. The cover of $\mathbb{G}^{\mathbf{2}}$ that we require is obtained by choosing for each $g \in G$ one of the open sets witnessing the fact that $g \in S_{sg}$.

First, to help us construct the arrows $\psi : U \to \mathbb{G}^{\nabla I}$, we describe how the object $\mathbb{G}^{\nabla I}$ admits a particularly simple description.

Lemma 5.1.2. *Let $\mathbb{G}$ be a groupoid in $\mathcal{E}$ with arrow space G, object space M and structure maps s, e, t, μ and i. Let X be an object of $\mathcal{E}$ and $a : 1 \to X$ be any global element of X. Then $Gpd_{\mathcal{E}}(\nabla X, \mathbb{G})$ is isomorphic to the subobject $G(X) \rightarrowtail G^X$ consisting of elements $\phi \in G^X$ such that the proposition*

$$(\phi(a) = es\phi(a)) \wedge (\forall x \in X.\ s\phi(x) = s\phi(a))$$

holds in the internal logic of $\mathcal{E}$.

Proof. First we note that $Gpd_{\mathcal{E}}(\nabla X, \mathbb{G})$ is the subobject of $G^{X\times X} \times M^X$ consisting of the pairs (ϕ_1, ϕ_0) such that the proposition

$$(s\phi_1 = \phi_0\pi_1) \wedge (t\phi_1 = \phi_0\pi_2) \wedge (\phi_1\Delta = e\phi_0) \wedge (\mu(2_\times\phi_1) = \phi_1(\pi_1, \pi_3))$$

holds in the internal logic. Then we define the arrows ξ_1 and ξ_2:

$$G(X) \underset{\xi_2}{\overset{\xi_1}{\rightleftarrows}} Gpd_{\mathcal{E}}(\nabla X, \mathbb{G})$$

as

$$\xi_1(\psi) = ((x, y) \mapsto \psi(y)\psi(x)^{-1}, t\psi)$$

(which is well typed because $s\psi$ is constant) and

$$\xi_2(\phi_1, \phi_0) = (x \mapsto \phi_1(a, x))$$

Now ξ_1 does factor through the appropriate subobject because the following equations hold:

$$\begin{aligned} s(\psi(y)\psi(x)^{-1}) &= t\psi(x) \\ t(\psi(y)\psi(x)^{-1}) &= t\psi(y) \\ \psi(x)\psi(x)^{-1} &= et\psi(x) \\ \mu(\psi(y)\psi(z)^{-1}, \psi(z)\psi(x)^{-1}) &= \psi(y)\psi(x)^{-1} \end{aligned}$$

and ξ_2 does factor through the corresponding subobject because the following equations

$$\phi_1(a, a) = e(\phi_0(a)) = e(\phi_0\pi_1(a, a)) = e(s\phi_1(a, a))$$

and

$$\forall x \in X.\ s\phi_1(a,x) = \phi_0\pi_1(a,x) = \phi_0(a) = \phi_0\pi_1(a,a) = s\phi_1(a,a)$$

hold. Now it remains to show that these two arrows are mutual inverses:

$$\begin{aligned} \xi_2(\xi_1(\psi)) &= \xi_2((x,y) \mapsto \psi(y)\psi(x)^{-1}, z \mapsto t(\psi(z))) \\ &= (u \mapsto \psi(u)\psi(a)^{-1}) \\ &= \psi \end{aligned}$$

where the last equality is due to the equation $\psi(a) = es\psi(a)$. In addition

$$\begin{aligned} \xi_1(\xi_2(\phi_1,\phi_0)) &= \xi_1(x \mapsto \phi_1(a,x)) \\ &= ((x,y) \mapsto \phi_1(a,y)\phi_1(a,x)^{-1}, z \mapsto t(\phi_1(a,z))) \\ &= ((x,y) \mapsto \phi_1(a,y)\phi_1(x,a), z \mapsto \phi_0(z)) \\ &= (\phi_1,\phi_0) \end{aligned}$$

where the last equality in the first factor is due to $\mu(2_\times\phi_1) = \phi_1(\pi_1,\pi_3)$. □

Notation 5.1.3. We will use the notation $C^k_\epsilon(z)$ for the cube $(-\epsilon,\epsilon)^k + z \subset \mathbb{R}^k$ that has sides of length $2\cdot\epsilon$ and is centred at z. In the case $\epsilon = 1$ we omit the subscript and write simply $C^k(z)$. In the case $z = 0$ we omit the brackets and write C^k_ϵ.

Definition 5.1.4. Let $s : G \to M$ be an arrow in Man and $x \in G$. Then a pair of open embeddings (ψ_x,ϕ_x) is an s-trivialisation at x iff

$$\begin{array}{ccc} C^{k+n} & \xrightarrow{\psi_x} & G \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle s} \\ C^k & \xrightarrow{\phi_x} & M \end{array}$$

commutes and $\psi_x(0) = x$.

Definition 5.1.5. A homotopy F from $\gamma : C \to G$ to $\gamma' : C \to G$ is an arrow in $\mathcal{E}$

$$C \xrightarrow{F} G^I$$

such that $G^0F = \gamma$ and $G^1F = \gamma'$. Note that this corresponds to a smooth map $C \times I \to G$ in Man.

Definition 5.1.6. An s-fibrewise homotopy F with respect to an arrow $s : G \to M$ is a homotopy such that

$$\begin{array}{ccccc} C & \xrightarrow{F} & G^I & \xrightarrow{G^0} & G \\ \downarrow{\scriptstyle F} & & & & \downarrow{\scriptstyle G^!} \\ G^I & \xrightarrow[s^I]{} & M^I & \xleftarrow[s^I]{} & G^I \end{array}$$

commutes.

Remark 5.1.7. If $\mathbb{G}$ is a Lie groupoid in Man then the source map $s : G \to M$ is (by definition) a submersion. This means that for all $x \in G$ we can find a trivialisation (ψ_x, ϕ_x).

Lemma 5.1.8. *Let $\mathbb{G}$ be a Lie groupoid with source map $s : G \to M$. Let $m \in M$. Then there is an s-trivialisation (ψ_{em}, ϕ_{em}) at em such that $e\phi_{em}$ factors through ψ_{em}.*

Proof. Let (ψ, ϕ) be any s-trivialisation at em. Then if ν and ξ are defined in the pullback

$$\begin{array}{ccc} P & \overset{\xi}{\rightarrowtail} & C^{k+n} \\ \downarrow{\scriptstyle \nu} & & \downarrow{\scriptstyle \psi} \\ C^k & \overset{e\phi}{\rightarrowtail} & G \end{array}$$

then $\phi\pi\xi = s\psi\xi = se\phi\nu = \phi\nu$ and so $\pi\xi = \nu$ because ϕ is a monomorphism. Now P is an open set of C^k and $0 \in P$ because $e\phi(0) = \psi(0)$. Since the derivative of ν has full rank at 0 we can find an open embedding $\iota : C^k \rightarrowtail P$ such that $\nu\iota(0) = 0$. Now let μ be defined by the pullback

$$\begin{array}{ccc} C^{k+n} & \overset{\mu}{\rightarrowtail} & C^{k+n} \\ {\scriptstyle \rho}\uparrow \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ C^k & \overset{\nu\iota}{\rightarrowtail} & C^k \end{array}$$

and ρ be induced by the pair $(1_P, \xi\iota)$. Then $e\phi\nu\iota = \psi\xi\iota = \psi\mu\rho$ and the s-trivialisation that we require is $(\psi_{em}, \phi_{em}) = (\psi\mu, \phi\nu\iota)$. □

Lemma 5.1.9. *For all $m \in M$ there is an s-trivialisation (ψ_{em}, ϕ_{em}) and an s-fibrewise homotopy*

$$F : C^{k+n} \to G^I$$

such that $G^0 \circ F$ factors through e and $G^1 \circ F = \psi_{em}$.

Proof. By Lemma 5.1.8 we can choose a s-trivialisation (ψ_{em}, ϕ_{em}) at em such that $e\phi_{em} = \psi_{em}\rho$ for some ρ. Now we define $r = C^{k+n} \to (C^{k+n})^I$ as

$$(z_1, z_2) \mapsto (u \mapsto (u(z_1, z_2) + (1-u)\rho(z_1)))$$

and then the F that we require is $F = \psi_{em}^I \circ r$. We see that F is s-fibrewise because (z_1, z_2) and $\rho(z)$ have the same first k coordinates. We confirm that $G^0 \circ F = \psi_{em}\rho\pi = e\phi_{em}\pi$ which factors through e and $G^1 \circ F = \psi_{em}$. □

Lemma 5.1.10. *Let (ψ_x, ϕ_x) be an s-trivialisation at some $x \in G$. Let $y = \psi((0, y_2))$ for some $y_2 \in C^n$ and (ψ_y, ϕ_y) be an s-trivialisation at y. Then there are open embeddings $\nu, \mu : C^{k+n} \rightarrowtail C^{k+n}$ and a s-fibrewise homotopy*

$$H : C^{k+n} \to G^I$$

such that $G^0 \circ H = \psi_y\nu$ and $G^1 \circ H = \psi_x\mu$.

Proof. First observe that $U_y = \psi_x^{-1}(\psi_y(C^{k+n}) \cap \psi_x(C^{k+n}))$ is an open set around $(0, y_2)$ in C^{n+k}. Therefore we can choose an ϵ such that $C_\epsilon^{k+n}(0, y_2)$ is contained in U_y; we see immediately that $C_\epsilon^{k+n}(0, 0)$ is contained in C^{k+n}. The μ that we require is the open embedding with image $C_\epsilon^{k+n}(0, 0)$. If η is the open embedding with image $C_\epsilon^{k+n}(0, y_2)$ then the ν that we require is $\psi_y^{-1}\psi_x\eta$.

Let $r : C^{k+n} \to (C^{k+n})^I$ be defined by

$$r(z) = (u \mapsto u\mu(z) + (1-u)\eta(z))$$

The homotopy H that we require is then $H = \psi_x^I r$. We see that H is s-fibrewise because the first k coordinates of ν and η coincide. Finally we confirm that $G^0 \circ H = \psi_x\eta = \psi_y\nu$ and $G^1 \circ H = \psi_x\mu$ as required. □

Proposition 5.1.11. *Let $\mathbb{G}$ be a Lie groupoid which is s-path connected as a topological space. Then the arrow $\mathbb{G}^l : \mathbb{G}^{\mathbb{I}} \to \mathbb{G}^{\mathbf{2}} = G$ is an epimorphism in $\mathcal{E}$.*

Proof. Using Lemma 5.1.2 and Proposition 5.1.1 it will suffice to find s-trivialisations (ψ_i, ϕ_i) such that the images of the ψ_i cover G together with s-fibrewise homotopies $F_i : C^{k+n} \to G^I$ such that $G^0 \circ F_i$ factors through $e : M \to G$ and $G^1 \circ F_i = \psi_i$.

To do this let S_m be the subset of $s^{-1}m$ consisting of all points x for which there exists a s-trivialisation (ψ_x, ϕ_x) at x and s-fibrewise homotopy $F : C^{k+n} \to G^I$ such that $G^0 \circ F$ factors through e and $G^1 \circ F = \psi_x$. We will show that for all $m \in M$ the sets S_m are closed, open and inhabited. Once we have established these three properties of S_m the proposition follows quickly because the hypothesis that $s^{-1}m$ is connected implies that $S_m = s^{-1}m$. Then we can use the cover $\bigcup_x im(\psi_x) = G$ where ψ_x is the first component of any s-trivialisation witnessing $x \in S_{sx}$.

To see that the set S_m is open let $x \in S_m$. This means that there exists an s-trivialisation (ψ_x, ϕ_x) and a s-fibrewise homotopy $K : C^{k+n} \to G^I$ such that $G^0 \circ K$ factors through e and $G^1 \circ K = \psi_x$. Now let y be any element in the open set $im(\psi_x) \cap s^{-1}m$. Using Lemma 5.1.10 we obtain open embeddings $\nu, \mu : C^{k+n} \rightarrowtail C^{k+n}$ and an s-fibrewise homotopy

$$H : C^{k+n} \to G^I$$

such that $G^0 \circ H = \psi_y \nu$ and $G^1 \circ H = \psi_x \mu$. Then by concatenating $K\mu$ with the reverse of H we see that $y \in S_m$.

To see that the set S_m is closed suppose $x \notin S_m$. Let U_x be any open set containing x and let (ψ_x, ϕ_x) be any s-trivialisation at x such that $im(\psi_x)$ is contained in U_x. Let $y \in im(\psi_x) \cap s^{-1}m$ and for the purpose of obtaining a contradiction suppose further that $y \in S_m$. This would mean that there exists a s-fibrewise homotopy

$$I : C^{k+n} \to G^I$$

such that $G^0 \circ I$ factors through e and $G^1 \circ I = \psi_y$. But now using Lemma 5.1.10 we would obtain open embeddings $\nu, \mu : C^{k+n} \rightarrowtail C^{k+n}$ and an s-fibrewise homotopy

$$H : C^{k+n} \to G^I$$

such that $G^0 \circ H = \psi_y \nu$ and $G^1 \circ H = \psi_x \mu$. Then by concatenating $I\nu$ with H we would see that $x \in S_m$. This contradiction shows that $im(\psi_x) \cap s^{-1}m$ is an open set containing x which is disjoint from S_m. Hence that the complement of S_m is open.

The set S_m is inhabited because $em \in S_m$ by Lemma 5.1.9. Hence $S_m = s^{-1}m$ and $\mathbb{G}^l$ is an epimorphism as required. □

5.2 Path Connectedness of Jet Part

As in the previous section we let $\mathcal{E}$ be any well-adapted model. We fix a Lie groupoid $\mathbb{G}$ in $\mathcal{E}$ that has arrow space G, object space M and structure maps s, t, e and μ. In Lemma 5.1.9 we have shown that for all $m \in M$ there is an s-trivialisation (ψ_{em}, ϕ_{em}) and a s-fibrewise homotopy

$$F : C^{k+n} \to G^I$$

such that $G^0 \circ F$ factors through e and $G^1 \circ F = \psi_{em}$. In this section we need to show that if we pull back ψ_{em} along ι_∞ to obtain a subobject $\tilde{\psi}_{em}$ of $\mathbb{G}_\infty$ we can choose F such that its restriction to $\tilde{\psi}_{em}$ factors through $\mathbb{G}_\infty$. To do this we will need to use Proposition 2.2.27 to show that a certain arrow is jet dense. In turn this means that we need to show that the relation $\approx$ is symmetric on the object (G, s) of $\mathcal{E}/M$. We first introduce the idea of a Mal'cev operation to show that $\approx$ is symmetric on C^n. This shows that every Lie groupoid $\mathbb{G}$ has an open cover such that for each element of this cover the relation $\approx$ is symmetric. Then we show that this does indeed imply that the relation $\approx$ is symmetric on (G, s).

Definition 5.2.1. Let B be an object of a finitely complete category. Then B admits a Mal'cev operation iff there exists an arrow $t : B^3 \to B$ such that $t(x, x, y) = y$ and $t(x, y, y) = x$.

Lemma 5.2.2. *For all n the space C^n admits a Mal'cev operation in Man.*

Proof. Let $a, b \in C^n$ and $U = \{(a, u, b) : a + b - u \notin C^n\} \subset C^{3n}$. Construct a smooth function $\phi : C^{3n} \to I$ which vanishes on the closed set U and attains the value 1 on the closed set$\{(a, u, b) : (u = a) \wedge (u = b)\}$. This is possible by Corollary 2.5 in [29]. Then the operation $t : C^{3n} \to C^n$ given by $(a, u, b) \mapsto \phi(a, u, b)(a + b - u)$ is Mal'cev. □

Lemma 5.2.3. *Let B be an object of $\mathcal{E}/M$ admitting a Mal'cev operation. Then $\sim$ and hence $\approx$ is symmetric on B in $\mathcal{E}/M$.*

Proof. Suppose that $a \sim b$. Then there exists $D_W \in Spec(Weil)$, $\phi \in B^{D_W}$ and $d \in D_W$ such that $\phi(0) = a$ and $\phi(d) = b$. Define $\psi \in B^{D_W}$ by $u \mapsto t(a, \phi(u), b)$.

Then $\psi(0) = t(a,a,b) = b$ and $\psi(d) = t(a,b,b) = a$ so we have that $b \sim a$ as required. □

Lemma 5.2.4. *Let $g : V \rightarrowtail G$ be a jet-closed arrow in $\mathcal{E}/M$ and suppose that the relation $\approx$ is symmetric on G. Then the relation $\approx$ is symmetric on V also.*

Proof. Let $a \approx b$ in V. Then by Remark 2.2.15 $ga \approx gb$ in G. Since $\approx$ is symmetric on G we see that $gb \approx ga$. But by Lemma 2.2.20 we have that $b \approx a$ as required. □

Lemma 5.2.5. *Let $(f_i : U_i \rightarrowtail B)_i$ be a cover of B in $\mathcal{E}/M$ such that each of the f_i is jet closed. Suppose that for all i the relation $\approx$ is symmetric on U_i. Then the relation $\approx$ is symmetric on B also.*

Proof. Suppose that $a \approx b$ in B. Since the f_i are jointly epimorphic there exist $i \in I$ and $x \in U_i$ such that $f_i(x) = a$. So there exists a $y \in U_i$ such that $f_i(y) = b$ because f_i is jet closed. Then Lemma 2.2.20 tells us that $x \approx y$ and hence $y \approx x$ because $\approx$ is symmetric on U_i. But then Remark 2.2.15 tell us that $b = f_i(y) \approx f_i(x) = a$ as required. □

Lemma 5.2.6. *Let $\mathbb{G}$ be a Lie groupoid and $(f_i : U_i \rightarrowtail G)_i$ be a cover of G in $\mathcal{E}$ such that for each i the open set U_i is trivial for s. Let g_i be the inclusions $s(U_i) \rightarrowtail M$. Then the relation $\approx$ is symmetric on $(U_i, g_i s)$ in $\mathcal{E}/M$.*

Proof. We will show that $(U_i, g_i s)$ admits a Mal'cev operation. The threefold product $(U_i, g_i s) \times (U_i, g_i s) \times (U_i, g_i s)$ in $\mathcal{E}/M$ is given by

$$B^3 = (U_i \,{}_s\!\times_s U_i \,{}_s\!\times_s U_i, g_i s \pi_1)$$

where π_1 is the projection from the limit onto the first factor. Since U_i is trivial for s we see that the arrow $s : U_i \to s(U_i)$ is isomorphic to $\pi : C^k \times C^n \twoheadrightarrow C^k$ in $\mathcal{E}$ via some ψ_i and ϕ_i. This means that

$$B^3 \cong (C^k \times C^{3n}, g_i \phi_i^{-1} \pi)$$

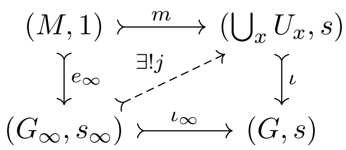
which admits a Mal'cev operation by Lemma 5.2.2. Therefore $(U_i, g_i s)$ admits a Mal'cev operation and so $\approx$ is symmetric on $(U_i, g_i s)$ as required. □

Corollary 5.2.7. *Let $\mathbb{G}$ be a Lie groupoid. Then the relation $\approx$ is symmetric on (G, s) in $\mathcal{E}/M$ (the object of arrows of $\mathbb{G}$).*

Proof. Follows immediately from combining Lemma 5.2.6 with Lemma 5.2.5. □

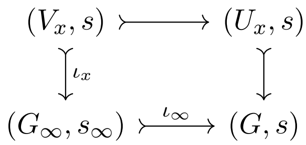
Lemma 5.2.8. *Let $(U_x \rightarrowtail G)_{x\in X}$ be a family of open subsets of G which cover $e(M)$ in $\mathcal{E}$. Let ι denote the inclusion $\bigcup_x U_x \rightarrowtail G$. Then there is an inclusion $j : G_\infty \rightarrowtail \bigcup_x U_x$ such that $\iota \circ j = \iota_\infty$.*

Proof. By hypothesis we have an inclusion $M \rightarrowtail \bigcup_x U_x$ such that $\iota \circ m = e$. Since the inclusion ι is jet closed in $\mathcal{E}/M$ the square

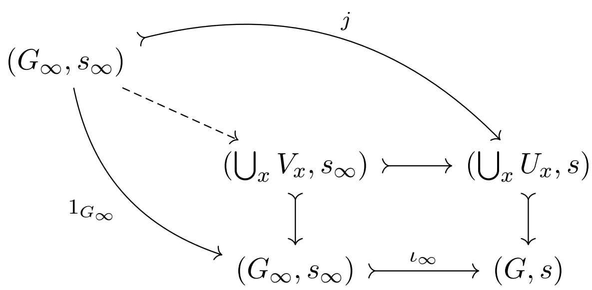


has a unique (monic) filler. □

Corollary 5.2.9. *Let V_x be defined by the pullback*



then because colimits are stable under pullback the bottom right square in



is a pullback. But then the arrow induced by the pair $(1_{G_\infty}, j) : G_\infty \to \bigcup_x V_x$ is an isomorphism and hence $(\iota_x : V_x \rightarrowtail G_\infty)_{x\in X}$ is a cover of G_∞.

Proposition 5.2.10. *Let $\mathbb{G}$ be a Lie groupoid and $\mathbb{G}_\infty$ its jet part. Then the $\mathbb{G}^l : \mathbb{G}_\infty^{\mathbb{I}} \to \mathbb{G}_\infty^{\mathbf{2}} = G_\infty$ is an epimorphism.*

Proof. We again use Proposition 5.1.1 to see that it will suffice to find a cover $(V_x \rightarrowtail G_\infty)_{x\in X}$ and s-fibrewise homotopies $K : V_x \to G^I_\infty$ such that $G^0_\infty \circ K$ factors through $e_\infty : M \rightarrowtail G_\infty$ and $G^1_\infty \circ K = \iota_x : V_x \rightarrowtail G_\infty$.

Let (ψ_{em}, ϕ_{em}) and F be the s-trivialisation and homotopy obtained in Lemma 5.1.9 and $U_{em} = im(\psi_{em})$. Thus we have that U_{em} is an open set that is trivial for s at x. Let V_{em} and W_{em} be defined by the iterated pullback:

$$\begin{array}{ccccc}
(W_{em}, \iota) & \overset{u_1}{\rightarrowtail} & (V_{em}, s_\infty) & \rightarrowtail & (U_{em}, s) \\
\downarrow & & \downarrow & & \downarrow \\
(M, 1) & \overset{e_\infty}{\rightarrowtail} & (G_\infty, s_\infty) & \overset{\iota_G}{\rightarrowtail} & (G, s)
\end{array}$$

Then by Proposition 2.2.27 and Lemma 5.2.4 we deduce that $u_1 : (W_{em}, \iota) \rightarrowtail (V_{em}, s_\infty)$ is jet dense and hence the arrow

$$1_{(I\times M,\pi_2)} \times u_1 : (I \times W_{em}, \iota \circ \pi_2) \to (I \times V_{em}, s_\infty \circ \pi_2)$$

is also jet dense. By Corollary 5.2.9 we have that $(V_{em} \rightarrowtail G_\infty)_{m\in M}$ is a cover of G_∞ and this is the one we will use to prove the proposition.

Now we observe that the restriction of F to W_{em} is constant at the inclusion $W_{em} \rightarrowtail G$:

$$F|_{W_{em}} = W_{em} \rightarrowtail G \xrightarrow{G^!} G^I$$

This means that the outer square of

$$\begin{array}{ccccc}
(I \times W_{em}, \iota \circ \pi_2) & \rightarrowtail & (I \times G_\infty, s_\infty \circ \pi_2) & \overset{\pi_2}{\longrightarrow} & (G_\infty, s_\infty) \\
\downarrow {\scriptstyle 1_{(I\times M,\pi_2)}\times u_1} & & {\scriptstyle \exists! \check{K}} & & \downarrow {\scriptstyle \iota_G} \\
(I \times V_{em}, s_\infty \circ \pi_2) & \rightarrowtail & (I \times U_{em}, s \circ \pi_2) & \overset{\check{F}}{\rightarrowtail} & (G, s)
\end{array}$$

commutes and so there exists a unique filler $\check{K}$ such that the transpose K is a s-fibrewise homotopy, $G^0_\infty \circ K$ factors through $e_\infty : M \rightarrowtail G_\infty$ and $G^1_\infty \circ K = \iota_{em} : V_{em} \rightarrowtail G_\infty$ as required. □

5.3 Simply Connectedness

At present I have not been able to establish that classical s-simply connected Lie groupoids are $\mathcal{E}$-simply connected. I suspect that the result can be established with similar arguments to those in Section 5.1 that prove that s-path connected

Lie groupoids are $\mathcal{E}$-path connected. The first additional difficulty that must be overcome to prove the simply connected version is to find some way of covering $\mathbb{G}^O$ in $\mathcal{E}$. The covers of G were in bijection with open covers of the underlying smooth manifold of G but the covers of $\mathbb{G}^O$ corresponding to pulling back covers of G along $\mathbb{G}^{\iota_1 l} : \mathbb{G}^O \to \mathbb{G}^{\mathbf{2}}$ are too coarse to be useful. Instead we will describe how to construct a cover of every generalised element of $\mathbb{G}^O$ using the compact open topology on the set $\Gamma(\mathbb{G}^O)$ of global sections of $\mathbb{G}^O$. We first relate the compact open topology on $\Gamma(\mathbb{G}^O)$ to a certain class of subobjects of $\mathbb{G}^O$ called Penon open subobjects. Then we relate Penon open subobjects of a representable object of $\mathcal{E}_{germ}$ with the Dubuc open sets of Definition 1.2.7. Having thus established a satisfactory way to cover generalised elements of $\mathbb{G}^O$ we proceed to prove that every simply connected Lie group $\mathbb{G}$ is $\mathcal{E}$-simply connected.

The second additional difficulty in proving the simply connected version is keeping track of the changes of chart. Intuitively speaking, the proof of the path connected version involved translating an open set in a source constant manner 'in parallel' to a given path. Since it was necessary to work in an s-trivialisation to make sense of how to move 'in parallel' we were required to change chart as we moved along the path. This was not too difficult to keep track of as we were only working in one dimension (the direction along the path). By contrast to prove the simply connected version it is necessary to translate an open set around a path in a source constant manner in parallel to a homotopy. This appears to require keeping track of s-trivialisations in two dimensions which I have not been able to successfully write down. Therefore we make the conjecture

Conjecture 5.3.1. *Every s-simply connected Lie groupoid $\mathbb{G}$ is $\mathcal{E}$-simply connected.*

which, if true, would imply that the definition of $\mathcal{E}$-simply connected groupoid is a legitimate generalisation of the definition of s-simply connected Lie groupoid. As mentioned above we can prove the special case of this conjecture when the base space $M = 1$ which corresponds to classical Lie theory.

First we turn to the problem of constructing an appropriate cover of $\mathbb{G}^O$.

Definition 5.3.2. A Penon open subobject is a monomorphism $m : A \rightarrowtail B$ in a topos $\mathcal{E}$ such that the proposition

$$\forall a \in A.\ \forall b \in B : (b \in A) \vee (\neg(ma = b))$$

holds in the internal logic of $\mathcal{E}$.

Lemma 5.3.3. *The collection of Penon open subobjects is stable under pullback. That is to say if m is a Penon open subobject and*

$$\begin{array}{ccc} C & \xrightarrow{g} & A \\ {\scriptstyle n}\downarrow & & \downarrow{\scriptstyle m} \\ D & \xrightarrow{f} & B \end{array}$$

is a pullback in $\mathcal{E}$ then n is a Penon open subobject.

Proof. The object C is carved out of D as the subobject

$$C = \{d \in D : fd \in A\}$$

Now let $c \in C$ and $d \in D$. Since m is a Penon open subobject we have that

$$(fd \in A) \vee (\neg(mgc = fd))$$

holds in the internal logic of $\mathcal{E}$. But $mgc = fnc$ and clearly

$$\neg(fnc = fd) \implies \neg(nc = d)$$

hence

$$(d \in C) \vee (\neg(nc = d))$$

as required. □

The following is Corollary 8 in [11].

Theorem 5.3.4. *Let $[n, I]$ be an object of $\mathcal{C}_{germ}$ and let $X \subset [n, I]$ be any subobject in $\mathcal{E}_{germ}$. Then X is Penon open iff it is of the form $X = \iota U \cap [n, I]$ for some $U \subset \mathbb{R}^n$ open.*

Corollary 5.3.5. *If $X \rightarrowtail [n, I]$ is Penon open then it is open in the sense of Definition 1.2.7.*

Next we recall some theory from [4] that relates the smooth compact open topology on global sections with Penon open subobjects. We first note that any function $f \in C^\infty(\mathbb{R}^p)$ can be thought of as a function $f \in C^\infty(\mathbb{R}^{p+n})$ by taking $f(x,t) = f(x)$ for all $t \in \mathbb{R}^n$.

Definition 5.3.6. An ideal $J \lhd C^\infty(\mathbb{R}^p)$ is said to have line determined extensions iff it satisfies the following condition: for every $n \in \mathbb{N}$ and $f \in C^\infty(\mathbb{R}^{p+n})$, $f \in J(x,t)$ iff for every fixed $a \in \mathbb{R}^n$, $f(x,a) \in J$.

Example 5.3.7. The ideal of all smooth functions vanishing on some closed subset of $\mathbb{R}^p$ has line determined extensions and hence by Remark 3.3.6 we see that the ideal defining the space Ω has line determined extensions.

Theorem 5.3.8. *Let $X = [p, J]$ and $Y = [n, I]$ in $\mathcal{C}_{germ}$ and assume that J has line determined extensions. Then the mapping $U \mapsto \Gamma(U)$ from the set of subobjects of Y^X to the set of subobjects of $\Gamma(Y^X)$ determines a bijection between the set of Penon open subobjects of Y^X and the set of smooth compact open subsets of $\Gamma(Y^X)$.*

Proof. Theorem 1.11 in [4]. □

Definition 5.3.9. The geodesic path between two points $x, y \in \mathbb{R}^n$ is the path defined by $a \mapsto (1-a)x + ay$. The geodesic homotopy between two paths $\gamma, \gamma' \in (\mathbb{R}^n)^I$ is the homotopy defined by $(a,b) \mapsto (1-a)\gamma(b) + a\gamma'(b)$.

Remark 5.3.10. We put the smooth compact open topology on the set $\Gamma(G^I)$. Then we give

$$\Gamma(G^I \times G^I) \cong \Gamma(G^I) \times \Gamma(G^I)$$

the induced product topology and

$$\Gamma(G^\Omega) \rightarrowtail \Gamma(G^I) \times \Gamma(G^I)$$

the induced subspace topology. Similarly for $\Gamma(G^B)$.

Proposition 5.3.11. *Let G be a simply connected Lie group. For all arrows $\phi : \Omega \to G$ in $\mathcal{E}$ such that $\phi(0) = e$ we can find a Penon open subobject*

$\Phi : V \to G^\Omega$ *containing* ϕ *such that for all* $v \in V$ *we have* $\Phi(v)(0) = e$ *and a smooth map* $\Psi : V \to G^B$ *such that*

$$\begin{array}{ccc} & & G^B \\ & \overset{\Psi}{\nearrow} & \big\downarrow{\scriptstyle G^\iota} \\ V & \xrightarrow{\Phi} & G^\Omega \end{array}$$

commutes.

Proof. First we fix a filler $H : B \to G$ of ϕ which we know exists because G is simply connected. Let $\psi : C^n \rightarrowtail G$ be an open embedding with $\psi(0) = e$ and let $im(\psi) = U$. Then we define the open set V as the set of all $\chi : \Omega \to G$ such that for all $a \in \Omega$ the element $\xi(a) = \phi(a)^{-1}\chi(a)$ is in U. In other words

$$V = \{\chi \in \Gamma(G^\Omega) : \forall a \in \Omega.\ \chi(a) \in \phi(a)U\}$$

and so is a well-defined open set in $\Gamma(G^U)$. By Theorem 5.3.8 the open set V is a Penon open subobject of G^Ω in $\mathcal{E}$. Now for all $\chi \in V$ the map $\xi : \Omega \to G$ lands entirely in U which is isomorphic to an open cube in $\mathbb{R}^n$. So we can define an arrow $g_1 : \Omega \to G$ as two copies of the geodesic path from e to $\xi(1)$. Note that $\xi(0) = e$ by construction. Define the arrow $g_2 : \Omega \to G$ by $g_2(a) = \phi(1)g_1(a)$. In addition we can find a geodesic homotopy

$$F : \mu(\xi, \delta\xi(1)) \Rightarrow \mu(\delta e, g_1)$$

where μ is the concatenation operation on paths and δx denotes the constant path at $x \in G$. Then the homotopy $F_2 = \mu(\phi, \delta\phi(1)) \cdot F$ is a homotopy between $\mu(\chi, \delta\chi(1))$ and $\mu(\phi, g_2)$ where $\cdot$ denotes pointwise multiplication of paths. Therefore to find a filler of $\mu(\chi, \delta\chi(1))$ it will suffice to find a filler of $\mu(\phi, g_2)$. But we see that $\mu_2(H, \delta g_2)$ is a filler of $\mu(\phi, g_2)$. By a straightforward reparametrisation we obtain a filler for χ. Now we remark that the filler ψ_χ varies smoothly with χ and so we obtain a smooth map $\Psi : V \to G^B$. □

Corollary 5.3.12. *Every simply connected Lie group* G *is* $\mathcal{E}_{germ}$*-simply connected.*

Proof. Let $\alpha : X \to G^\Omega$ be an arrow such that X is a representable object in $\mathcal{E}_{germ}$ and $G^0\alpha$ factors through e. By Proposition 5.1.1 it will suffice to find a cover $(\iota_i : X_i \to X)_{i \in I}$ such that for all i there exists $\beta : X_i \to G^B$ such that $G^\iota \beta = \alpha \iota_i$ and for all $x_i \in X_i$ we have that $\beta(x_i)(0,0) = e$. Now for every global element $x \in_1 X$ we obtain a global element $\phi = \alpha x \in_1 G^\Omega$ such that $\phi(0) = e$. Hence by Proposition 5.3.11 above there exists a Penon open subobject $\Phi : V \rightarrowtail G^\Omega$ containing ϕ such that for all $v \in V$ we have $\Phi(v)(0) = e$ and a smooth map $\Psi : V \to G^B$ such that the top triangle in

$$\begin{array}{ccc} & & G^B \\ & \overset{\Psi}{\nearrow} & \downarrow G^\iota \\ V & \overset{\Phi}{\rightarrowtail} & G^\Omega \\ \alpha_x \uparrow & & \uparrow \alpha \\ W_x & \overset{n_x}{\rightarrowtail} & X \end{array}$$

commutes and for all $v \in V$ we have $\Psi(v)(0) = e$. Then if we pullback Φ along α we obtain by combining Corollary 5.3.5 and Lemma 5.3.3 an open subobject $n_x : W_x \rightarrowtail X$ that contains x. Then the cover that we require is $(n_x : W_x \to X)_{x \in_1 X}$ and the lift $\beta = \Psi \alpha_x$. By construction we have that $G^\iota \Psi \alpha_x = \alpha n_x$ and for all $w \in W_x$ we have that $\Psi \alpha_x w(0,0) = e$ as required. □

5.4 Integral Completion

Recall that our motivation for introducing the integral factorisation system in Section 2.3.3 was to provide in the topos $\mathcal{E}$ an axiomatic alternative to solving time-dependent left-invariant vector fields on a Lie groupoid. A groupoid $\mathbb{X}$ in which it is always possible to solve such vector fields satisfies the condition that $\mathbb{X}^{\nabla I_\infty} \cong \mathbb{X}^{\nabla I}$ in $Gpd(\mathcal{E})$. Indeed this is an assumption that we required in our proof of Lie's second theorem. In the first part of this section we relate the global elements of each of the objects $\mathbb{X}^{\nabla I_\infty}$ and $\mathbb{X}^{\nabla I}$ to established structures in classical Lie theory. The global elements of $\mathbb{X}^{\nabla I}$ are straightforwardly seen to be the $\mathbb{G}$-paths that can be found in for example [7]. The only complication in identifying global elements of $\mathbb{X}^{\nabla I_\infty}$ with the A-paths found in [7] is that Lie algebroids are only concerned with linear infinitesimals and ∇I_∞ involves arbitrary jets. Recall that ∇I_∞ has as its arrow space the subspace of I^2 consisting of all the pairs $(a, b) \in I^2$ such that $a \approx b$. It turns out however that there is a way of formally integrating vector fields in synthetic differential geometry for a certain well-behaved type of object of $\mathcal{E}$ called a microlinear space (of which all manifolds are examples). The idea (which can be found for example in [20]) is that an action of the additive group D_∞ is completely determined by the subobject D. In order to use this idea we need to first need to make explicit the relationship between groupoid homomorphisms $\nabla I_\infty \to \mathbb{G}$ and actions of the additive group D_∞.

Definition 5.4.1. The object $A(\mathbb{G})$ of A-paths associated to a Lie groupoid $\mathbb{G}$ is the subobject of $G^{I\times D} \times M^I$ consisting of all the pairs $(\phi, \gamma) \in G^{I\times D} \times M^I$ such that the proposition

$$\forall a \in I.\ \forall d \in D.\ (s\phi = \gamma) \wedge (t\phi(a,d) = \gamma(a+d)) \wedge (\phi(a,0) = es\phi(a,0))$$

holds in the internal logic of $\mathcal{E}$. A classical A-path is just a global element of the object of A-paths in $\mathcal{E}$.

Remark 5.4.2. We can rephrase this definition in terms of reflexive graphs in $\mathcal{E}$. Indeed

$$A(\mathbb{G}) \cong Refl_{\mathcal{E}}(I_D, u\mathbb{G})$$

where I_D is the reflexive graph

$$I_D = I \times D \underset{\pi_1}{\overset{+}{\underset{\longleftarrow e -}{\rightrightarrows}}} I$$

in $\mathcal{E}$, $Refl_{\mathcal{E}}$ denotes the $\mathcal{E}$-valued hom of reflexive graphs internal to $\mathcal{E}$ and $u\mathbb{G}$ is the underlying reflexive graph of $\mathbb{G}$.

Remark 5.4.3. Since $G^D \cong TG$ the classical A-paths are precisely the A-paths described in Section 1.1 of [7].

Definition 5.4.4. A microlinear space M in $\mathcal{E}$ is an object of $\mathcal{E}$ which satisfies the following property. If $D : \mathcal{J} \to \mathcal{E}$ is a diagram in $\mathcal{E}$ such that for all objects j of $\mathcal{J}$ the object $D(j)$ is the spectrum of a Weil algebra and

$$lim_D R^{D(j)} \cong R^{D_W}$$

in $\mathcal{E}$ for some Weil spectrum D_W then

$$lim_D M^{D(j)} \cong M^{D_W}$$

in $\mathcal{E}$ also.

Example 5.4.5. All manifolds in the image of the full and faithful embedding $\iota : Man \to \mathcal{E}$ are microlinear.

Lemma 5.4.6. *Let $k \in \mathbb{N}$ and consider the arrows $f_1, ..., f_k : D^{k-1} \to D^k$ defined by*

$$f_i(d_1, ..., d_k) = (d_1, ..., d_{i-1}, 0, d_i, ..., d_{k-1})$$

Then for any microlinear space G the arrow

$$G^{D_k} \xrightarrow{G^+} G^{D^k}$$

is the joint equaliser of f_1,...,f_k.

Proof. Let $\phi \in R^{D^k}$ such that $R^{f_i}\phi = R^{f_j}\phi$ for all $i, j \in \{1, ..., k\}$. In particular this means that $R^{u_i}\phi = R^{u_j}\phi$ where $u_i : D \to D^k$ is the inclusion $d \mapsto (0, ..., 0, d, 0, ..., 0)$ in which the non-zero entry in the k-tuple is in the ith position. Now using the Kock-Lawvere axiom we deduce that ϕ is of the form

$$\phi(d_1, ..., d_k) = \Sigma_{i=0}^{k} a_i \cdot e_i(d_1, ..., d_k)$$

where $a_i \in \mathbb{R}$ and e_i is the ith symmetric polynomial

$$e_i(d_1, ..., d_k) = \Sigma_{1 \leq j_1 < j_2 < ... < j_i \leq k} d_{j_1} d_{j_2} ... d_{j_i}$$

Now the result follows from the fact that $(d_1 + d_2 + ... + d_k)^i$ is a multiple of $e_i(d_1, d_2, ..., d_k)$ whenever $d_j^2 = 0$ for all j. □

The following proposition is a generalisation of Remark 3.10 in Chapter V of [28].

Proposition 5.4.7. *The arrow*

$$\mathbb{G}^{\nabla I_\infty} \xrightarrow{G^{\iota_\infty \times I}} Refl_{\mathcal{E}}(I_D, u\mathbb{G})$$

is an isomorphism.

Proof. We construct an inverse ν for $G^{\iota_\infty \times I}$. So let $(\phi, \xi) \in Refl_{\mathcal{E}}(I_D, u\mathbb{G})$. Then the arrow $\eta_k : G^D \to G^{D^k}$ defined by

$$(d_1, d_2, ..., d_k, \phi) \mapsto \phi(a + \Sigma_{i=1}^{n-1} d_i, d_n) ... \phi(a + d_1, d_2)\phi(a, d_1)$$

satisfies the condition $G^{f_i}\eta_k = G^{f_j}\eta_k$ for all $i, j \in \{1, ..., k\}$ for the f_i defined in Lemma 5.4.6. Hence we have that η_k factors through G^{D_k} as $\nu_k : G^D \to G^{D_k}$. Since $D_\infty = \bigcup_k D_k$ and

$$v_k(\phi)(d_1, ..., d_{k-1}, 0) = v_{k-1}(\phi)(d_1, ..., d_{k-1})$$

the family $(\nu_k)_{k\in\mathbb{N}_{>0}}$ induces an arrow $\nu : G^D \to G^{D_\infty}$. Now since ϕ is a reflexive graph homomorphism $\nu(\phi)$ is also. We have that $\nu(\phi)$ preserves composition by construction. □

Corollary 5.4.8. *We have an isomorphism $A(\mathbb{G}) \cong \mathbb{G}^{\nabla I_\infty}$.*

Definition 5.4.9. The object $Path(\mathbb{G})$ of $\mathbb{G}$-paths of a Lie groupoid $\mathbb{G}$ is the object $\mathbb{G}^{\nabla I}$ in $\mathcal{E}$. A classical $\mathbb{G}$-path is a global element of $\mathbb{G}^{\nabla I}$ i.e. an arrow $\nabla I \to \mathbb{G}$.

Remark 5.4.10. By Lemma 5.1.2 the object of $\mathbb{G}$-paths is the same as the subobject of G^I consisting of all the elements $\psi \in G^I$ such that the proposition

$$(\psi(0) = es\psi(0)) \wedge (\forall a \in I.\ s\psi(a) = s\psi(0))$$

and hence our classical $\mathbb{G}$-paths are precisely the $\mathbb{G}$-paths described in the introduction of [7].

Lemma 5.4.11. *For every groupoid $\mathbb{G}$ in $\mathcal{E}$ and $D_W \in Spec(Weil)$ we have a groupoid $\mathbb{G}^{D_W}$ that has underlying reflexive graph*

$$G^{D_W} \underset{t^{D_W}}{\overset{s^{D_W}}{\rightrightarrows}} M^{D_W} \quad (e^{D_W} : M^{D_W} \to G^{D_W})$$

composition μ^{D_W} and inverse $i_{\mathbb{G}}^{D_W}$.

Lemma 5.4.12. *For all manifolds M and Weil presentations W the object M^{D_W} is also a manifold. The object M^{D_W} is the image under the full and faithful embedding $\iota : Man \to \mathcal{E}$ of $T_W M$ which is called (if we pair it with the projection $M^0 : M^{D_W} \to M$) the Weil bundle of M with respect to W.*

Proof. We refer to Theorems 35.13 and 35.14 in [21]. □

The next Lemma is a sketch of the proof that Proposition 1.1 in [7] is unaffected by smooth dependence on a parameter.

Lemma 5.4.13. *Let $\mathbb{G}$ be a groupoid and X a smooth manifold. Then there is a bijection between the set S_1 of smooth maps*

$$X \times I \xrightarrow{(F_1,F_2)} TG \times M$$

in Man such that

$$\frac{d(sF_1(x,a))}{da} = 0,\ \frac{d(tF_1(x,a))}{da} = \frac{dF_0(x,a)}{da} \text{ and } \pi F_1(x,a) = eF_0(x,a)$$

hold and the set S_2 of smooth maps

$$X \times I \xrightarrow{\psi} G$$

in Man such that

$$\psi(x,0) = es\psi(x,0) \text{ and } \forall a \in I.\ s\psi(x,a) = s\psi(x,0)$$

hold.

Proof. (Sketch) Let $F \in S_1$. First we extend $F_1(x,a)$ to any map $\alpha : X \times I \times M \to TG$ such that $\alpha(x,a,F_0(x,a)) = F_1(x,a)$. Then define $\Phi : S_1 \to S_2$ by sending F to the unique solution $\psi_F : X \times I \to G$ of the differential equation

$$\frac{d\psi_F(x,a)}{da} = (DR_{\psi_F(x,a)})\alpha(x,a,t\psi_F(x,a)) \tag{5.1}$$

with initial condition $\psi(x,0) = F_0(0)$ where R_g denotes precomposition with g and D denotes the derivative. Note that because the derivative of sF_1 with respect to a is 0 the solution ψ has constant source. In the other direction define $\Psi : S_2 \to S_1$ as taking

$$\psi \mapsto \left((DR_{\psi(x,a)^{-1}})\frac{\psi(x,a)}{da}, t\psi(x,a)\right)$$

where we note that the three conditions defining S_1 hold by construction. Now $\Phi\Psi$ is the identity by the uniqueness of the solution ψ. In addition we calculate

$$\begin{aligned}\Psi\Phi(F(x,a)) &= \left((DR_{\psi_F}\frac{d\psi_F(x,a)}{da}), t\psi_F(x,a)\right)\\ &= (\alpha(x,a,t\psi_F(a)), t\psi_F(x,a))\\ &= (\alpha(x,a,F_1(x,a), F_0(x,a)) = F(x,a)\end{aligned}$$

as required. □

Proposition 5.4.14. *For every Lie groupoid $\mathbb{G}$ the arrow $\mathbb{G}^{\nabla I_\infty} \xrightarrow{\mathbb{G}^{\iota_\infty}} \mathbb{G}^{\nabla I}$ is an isomorphism in the Cahiers topos.*

Proof. It will suffice to find for all representable $X \times D_W$ in the Cahiers topos an inverse to the component $\mathbb{G}^{\iota_\infty}_X$. Furthermore since $\mathbb{G}^{D_W}$ is a Lie groupoid for all $D_W \in Spec(Weil)$ it will suffice to consider generalised elements with domain X. We use the identification $\mathbb{G}^{\nabla I_\infty} \cong A(\mathbb{G})$ established in Corollary 5.4.8. So let $\phi \in_X A(\mathbb{G})$. Then we have the bijections

$$\begin{array}{ll} X \xrightarrow{\phi} A(\mathbb{G}) & \text{in } \mathcal{E} \\ \hline X \to G^{I \times D} \times M^I & \text{in } \mathcal{E} \\ \hline X \times I \xrightarrow{\tilde{F}} G^D \times M & \text{in } \mathcal{E} \\ \hline X \times I \xrightarrow{F} TG \times M & \text{in } Man \end{array}$$

where F satisfies the conditions

$$\frac{d(sF_1(x,a))}{da} = 0, \ \frac{d(tF_1(x,a))}{da} = \frac{dF_0(x,a)}{da} \text{ and } \pi F_1 = eF_0$$

and we have neglected to state the conditions we should impose on the second line. So then we use Lemma 5.4.13 to deduce that we have a bijection

$$\begin{array}{c} X \xrightarrow{\phi} A(\mathbb{G}) \\ \hline X \times I \xrightarrow{\psi_\phi} G \end{array}$$

such that

$$\psi_\phi(x,0) = es\psi_\phi(x,0) \text{ and } \forall a \in I.\ s\psi_\phi(x,a) = s\psi_\phi(x,0)$$

hold. But this is precisely what it means to be a generalised element of $\mathbb{G}^{\nabla I}$ at stage of definition X. It remains to check that

$$\begin{aligned} \tilde{F}_1(x,a)(d) &= \psi_\phi(x,a+d)\psi_\phi(x,a)^{-1} \\ &= e + d\frac{\partial \psi_\phi(x,a)}{\partial a}\psi_\phi(x,a)^{-1} \end{aligned}$$

which is equivalent to

$$F_1(x,a) = (DR_{\psi_\phi(x,a)^{-1}})\frac{\partial \psi_\phi(x,a)}{\partial a}$$

which holds by the construction of ψ_ϕ in Lemma 5.4.13. □

Now that we have established that $\mathbb{G}^{\iota_\infty}$ is an isomorphism in the Cahiers topos it remains to prove the following Proposition.

Proposition 5.4.15. *If* $\mathbb{G}^{\iota_\infty} : \mathbb{G}^{\nabla I} \to \mathbb{G}^{\nabla I_\infty}$ *is an isomorphism in a well-adapted model* $\mathcal{E}$ *then it is an isomorphism of groupoids also.*

Proof. We need to show that natural transformations extend uniquely i.e.:

$$\begin{array}{ccc} \nabla I_\infty \times \mathbf{2} & \xrightarrow{\forall \Phi} & \mathbb{G} \\ {\scriptstyle \iota}\downarrow & \nearrow_{\exists ! \Psi} & \\ \nabla I \times \mathbf{2} & & \end{array}$$

Let ψ_0, ψ_1 be the unique lifts of ϕ precomposed with the two inclusions of 1 into $\mathbf{2}$. If for all $x \to y$ in ∇I the diagram

$$\begin{array}{ccc} \Phi(x,1) & \xrightarrow{\psi_1(x \to y)} & \Phi(y,1) \\ {\scriptstyle \Phi(x,0\to 1)}\uparrow & & \uparrow{\scriptstyle \Phi(y,0\to 1)} \\ \Phi(x,0) & \xrightarrow{\psi_0(x \to y)} & \Phi(y,0) \end{array} \tag{5.2}$$

commutes then we can define $\Psi(x \to y, 0 \to 1)$ to be this common value. To this end define $\theta : \nabla I \to M$ to take $x \to y$ to:

$$\begin{array}{ccc} \Phi(x,1) & & \Phi(y,1) \\ {\scriptstyle \Phi(x,1\to 0)}\downarrow & & \uparrow{\scriptstyle \Phi(y,0\to 1)} \\ \Phi(x,0) & \xrightarrow{\psi_0(x \to y)} & \Phi(y,0) \end{array}$$

when we restrict to ∇I_∞ (i.e. take $y = x + d$) we see that:

$$\begin{array}{ccc} \Phi(x,1) & & \Phi(x+d,1) \\ {\scriptstyle \Phi(x,1\to 0)}\downarrow & & \uparrow{\scriptstyle \Phi(x+d,0\to 1)} \\ \Phi(x,0) & \xrightarrow{\Phi(x \to x+d,0)} & \Phi(x+d,0) \end{array} = \Phi(x,1) \xrightarrow{\Phi(x \to x+d,1)} \Phi(x+d,1)$$

and so by the uniqueness of lifts $\theta = \psi_1$ and so the diagram (5.2) commutes. $\square$

Conclusion

In this thesis we have constructed an adjunction

$$Cat_\infty(\mathcal{E}) \underset{(-)_\infty}{\overset{(-)_{int}}{\rightleftarrows}} Cat_{int}(\mathcal{E}) \qquad \perp \tag{5.3}$$

which generalises the classical Lie adjunction between the category of formal group laws and the category of Lie groups. Moreover the category of local objects $Cat_\infty(\mathcal{E})$ and the category of global objects $Cat_{int}(\mathcal{E})$ are coreflective and reflective subcategories respectively of the larger category $Cat(\mathcal{E})$. We showed that when we restrict the domain of $(-)_\infty$ to the integral complete and $\mathcal{E}$-simply connected categories that have $\mathcal{E}$-path connected jet part then $(-)_\infty$ becomes full and faithful. We also showed how to modify this theory to create an adjunction involving groupoids rather than categories and made the first steps towards a proof that all source simply connected Lie groupoids are $\mathcal{E}$-simply connected, integral complete and have $\mathcal{E}$-path connected jet part.

A natural goal for future research would be to complete the description of the relationship between the groupoid version of Diagram 5.3 and the adjunction underlying classical multi-object Lie theory. This would involve proving that all source simply connected Lie groupoids are $\mathcal{E}$-simply connected and removing the dependence on special properties of the Cahiers topos from the proof that all Lie groupoids are integral complete. In addition the relationship between the jet part $\mathbb{G}_\infty$ of a Lie groupoid $\mathbb{G}$ and the Lie algebroid $\mathfrak{g}$ of $\mathbb{G}$ could be investigated. In fact the natural counterpart with which to compare $\mathbb{G}_\infty$ would be a multi-object generalisation of a formal group law and hence it would be interesting to compare $\mathbb{G}_\infty$ to the local objects constructed in [9] and [30]. One could similarly try to find a classical counterpart to the symmetric jet part construction which is possible in the Cahiers topos.

For future research in the synthetic theory itself it would be logical to look for a characterisation of the jet categories for which the unit of the adjunction in Diagram 5.3 is an isomorphism. For more detail see Section 4.2. Finally since the theory of integrating Lie algebroids has application in the area of Poisson geometry it would be interesting to study this formulation of Hamiltonian mechanics in terms of infinitesimals. A first step in this direction might be a treatment of the correspondence between Poisson manifolds and symplectic groupoids (that can be found in for example [36]) using synthetic differential geometry.

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