Synthetic Lie Theory

Matthew Burke, Macquarie University

CT 2014, University of Cambridge

Outline

1. The classical integrability problem for Lie Algebroids
2. The integrability problem in synthetic differential geometry
3. Lie's second theorem in synthetic differential geometry
4. The Weinstein groupoid and discussion of Lie III

Classical Lie Theory

Recall that there is a functor

$$LieGp_{sc} \xrightarrow{T_e} LieAlg$$

that takes a simply connected Lie group to its Lie algebra.

- Lie's second theorem says that T_e is full and faithful.
- Lie's third theorem says that T_e is essentially surjective.

Definition 1.
A *Lie groupoid* is a groupoid in *Man* such that the source and target maps are submersions.

Definition 2.
A *Lie algebroid* is a vector bundle $A \to M$ in *Man* together with a bundle homomorphism $\rho : A \to TM$ such that the space of sections $\Gamma(A)$ is a Lie algebra satisfying $(\forall X, Y \in \Gamma(A))(\forall f \in C^\infty(M))$:

$$[X, fY] = \rho(X)(f) \cdot Y + f[X, Y]$$

Classical Lie Theory

In the multiobject setting, we still have a full and faithful functor

$$LieGpd_{sc} \xrightarrow{T_e} LieAlgd$$

but it is not essentially surjective.

- For every Lie algebroid there is a topological groupoid that is the ‘obvious’ candidate for the integral of the algebroid (its Weinstein groupoid) but there can be obstructions to putting a smooth structure on it - see [Crainic and Fernandes 2003].

Idea: Enlarge the category of smooth spaces:-

- Differentiable Stacks [Tseng and Zhu 2006].
- Using Synthetic Differential Geometry.

Relevant Features of Synthetic Differential Geometry

We work in the Grothendieck topos $\mathcal{C}$ called the *Cahiers* topos that is a well-adapted model of synthetic differential geometry. This means that:-

- There is a full and faithful embedding $Man \xrightarrow{\iota} \mathcal{C}$.
- There is a ring $R = \iota(\mathbb{R}) \in \mathcal{C}$.
- The object $D_k = \{x \in R : x^{k+1} = 0\}$ is not terminal, in fact the Kock-Lawvere axiom holds:

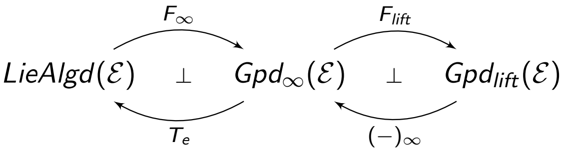
$$R^{k+1} \to R^{D_k}$$
$$(a_0, a_1, .., a_k) \mapsto (d \mapsto a_0 + a_1 d + ... + a_k d^k)$$

  is an isomorphism.

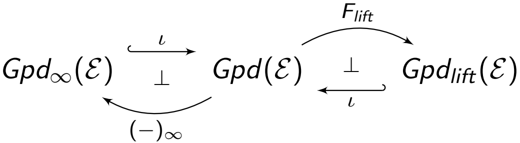
Let $\mathcal{E}$ be the full subcategory of *microlinear objects* of $\mathcal{C}$.

Big Picture

Now that we have infinitesimal objects we can use an intermediary category:



where the right hand adjunction is the composite of



where $Gpd_\infty(\mathcal{E})$ and $Gpd_{lift}(\mathcal{E})$ are categories of (co-)fibrant objects with respect to factorisation systems to be defined later.

The Jet Factorisation System

Definition 3.

A *jet-closed arrow in $\mathcal{E}$* is a monic arrow $m : A \rightarrowtail B$ such that the following square is a pullback:

$$\begin{array}{ccc} A^D & \xrightarrow{-\circ 0} & A \\ \downarrow{\scriptstyle m\circ -} & & \downarrow{\scriptstyle m\circ -} \\ B^D & \xrightarrow{-\circ 0} & B \end{array}$$

Definition 4.

A *jet-dense arrow in $\mathcal{E}$* is an arrow $f : X \to Y$ such that the following square is a pullback for every jet-closed m:

$$\begin{array}{ccc} A^Y & \xrightarrow{-\circ f} & A^X \\ \downarrow{\scriptstyle m\circ -} & & \downarrow{\scriptstyle m\circ -} \\ B^Y & \xrightarrow{-\circ f} & B^X \end{array}$$

The Jet Factorisation System

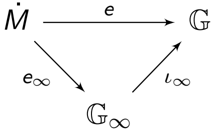
Definition/Proposition 5.

The jet factorisation system on $\mathcal{E}$ is given by

$$(L, R) = (\textit{jet-dense}, \textit{jet-closed})$$

and it induces a factorisation system on $Gpd(\mathcal{E})$.

Now for any groupoid $\mathbb{G} = (G \rightrightarrows M) \in Gpd(\mathcal{E})$ we can factorise the identity arrow:



If $Gpd_\infty(\mathcal{E})$ is the subcategory of groupoids which have jet-dense identity map then the above factorisation induces a functor:

$$(-)_\infty : Gpd(\mathcal{E}) \to Gpd_\infty(\mathcal{E})$$

Some Useful Groupoids

$\mathbb{2}$ is the pair groupoid on the two element space.
$\mathbb{I}$ is the pair groupoid on the unit interval.
ℓ is the long arrow $0 \to 1$ in $\mathbb{I}$.
O is the pushout of ℓ along itself:

$$\begin{array}{ccc} \mathbb{2} & \xrightarrow{\ell} & \mathbb{I} \\ \downarrow{\scriptstyle \ell} & & \downarrow{\scriptstyle \iota_0} \\ \mathbb{I} & \xrightarrow{\iota_1} & O \end{array}$$

$\mathbb{O}$ is the colimit of the diagram:

$$1 \longleftarrow \mathbb{I} \xrightarrow{1\times\{0\}} \mathbb{I}\times\mathbb{I} \xleftarrow{1\times\{1\}} \mathbb{I} \longrightarrow 1$$

$\iota : O \to \mathbb{O}$ is the boundary inclusion. Let $\mathbb{I}_\infty$, O_∞ and $\mathbb{O}_\infty$ be the images of $\mathbb{I}$, O and $\mathbb{O}$ under $(-)_\infty$.

Connectedness of Groupoids

Definition 6.

A *path-connected groupoid* is a groupoid $\mathbb{G}$ for which the following arrow is an epimorhpism:

$$\mathbb{G}^{\mathbb{I}} \xrightarrow{-\circ \ell} \mathbb{G}^{\mathbb{2}}$$

A *simply-connected groupoid* is a path-connected groupoid $\mathbb{G}$ for which the following arrow is an epimorphism:

$$\mathbb{G}^{\mathbb{O}} \xrightarrow{-\circ \iota} \mathbb{G}^{O}$$

Proposition 7.

If $\mathbb{G}$ is simply connected then the following diagram is a coequaliser in $\mathcal{E}$:

$$\mathbb{G}^{\mathbb{O}} \underset{-\circ(\iota\circ\iota_1)}{\overset{-\circ(\iota\circ\iota_0)}{\rightrightarrows}} \mathbb{G}^{\mathbb{I}} \xrightarrow{-\circ \ell} \mathbb{G}^{\mathbb{2}}$$

The Lifting Property for Integration

In the synthetic setting we cannot assume that we always have solutions to time-dependent invariant vector fields.

Definition 8.

A *groupoid with the lifting property* is a groupoid $\mathbb{G}$ for which the following arrow is an isomorphism

$$\mathbb{G}^{\mathbb{I}} \xrightarrow{-\circ\iota_\infty} \mathbb{G}^{\mathbb{I}_\infty}$$

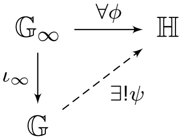
Proposition 9.

If $\mathbb{G}$ has the lifting property then $-\circ\iota_\infty$ is also an isomorphism of groupoids. Moreover, $\mathbb{G}^{\mathbb{O}} \cong \mathbb{G}^{\mathbb{O}_\infty}$.

Idea of Proof: Show that the commutativity of infinitesimal squares extends to the commutativity of macroscopic squares.

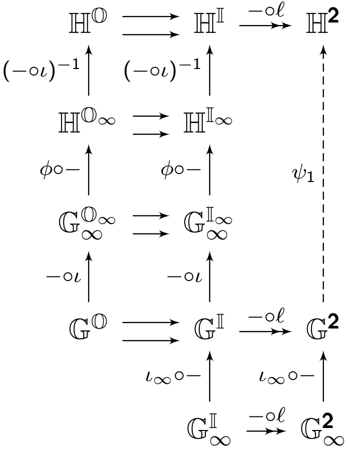
Synthetic Lie II

Theorem 10.
Let $\mathbb{G}$ be a simply-connected groupoid and let $\mathbb{H}$ be a groupoid with the lifting property. Then the following lifting property holds:



Proof: On the next slide.

Synthetic Lie II



- We take the object map $\psi_0 = \phi_0$.
- For the arrow map we consider the diagram opposite. The penultimate row is a coequaliser because $\mathbb{G}$ is simply-connected. The maps $(- \circ \iota)^{-1}$ exist because $\mathbb{H}$ has the lifting property.

The Weinstein Groupoid

Definition 11.

The Weinstein groupoid $\bar{\mathbb{G}} = (\bar{G} \rightrightarrows M)$ of the groupoid $\mathbb{G} = (G \rightrightarrows M)$ has:

- object space M.
- arrow space the coequaliser:

$$\mathbb{G}^{\mathbb{O}_\infty} \underset{-\circ(\iota\circ\iota_1)}{\overset{-\circ(\iota\circ\iota_0)}{\rightrightarrows}} \mathbb{G}^{\mathbb{I}_\infty} \xrightarrow{q} \bar{\mathbb{G}}^2$$

- reflexive graph structure and multiplication induced by the factorisations of $ev_0, ev_1, const$ and μ. For example:

$$\begin{array}{ccccc}
\mathbb{G}^{\mathbb{O}_\infty +_* \mathbb{O}_\infty} & \underset{-\circ(\iota\circ\iota_1)}{\overset{-\circ(\iota\circ\iota_0)}{\rightrightarrows}} & \mathbb{G}^{\mathbb{I}_\infty +_* \mathbb{I}_\infty} & \longrightarrow & \bar{\mathbb{G}}^2 \times_{G_0} \bar{\mathbb{G}}^2 \\
\downarrow \mu & & \downarrow \mu & & \downarrow \mu \\
\mathbb{G}^{\mathbb{O}_\infty} & \underset{-\circ(\iota\circ\iota_1)}{\overset{-\circ(\iota\circ\iota_0)}{\rightrightarrows}} & \mathbb{G}^{\mathbb{I}_\infty} & \longrightarrow & \bar{\mathbb{G}}^2
\end{array}$$

References

Marius Crainic and Rui Loja Fernandes.
Integrability of Lie brackets.
Ann. of Math. (2), 157(2):575–620, 2003.

Anders Kock.
Synthetic differential geometry, volume 333 of *London Mathematical Society Lecture Note Series*.
Cambridge University Press, Cambridge, second edition, 2006.

Ieke Moerdijk and Gonzalo E. Reyes.
Models for smooth infinitesimal analysis.
Springer-Verlag, New York, 1991.

Hsian-Hua Tseng and Chenchang Zhu.
Integrating Lie algebroids via stacks.
Compos. Math., 142(1):251–270, 2006.