Synthetic Lie Theory

Matthew Burke

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Outline

Classical Lie Theory

Recall that in classical Lie theory we have an adjunction

$$
\mathit{LieAlg} \underset{(-)_{\infty}}{\overset{(-)_{int}}{\rightleftarrows}} \mathit{LieGrp} \quad (\bot)
$$

When we restrict to the simply connected Lie groups:

- Lie's second theorem says that $(-)_{\infty}$ is full and faithful
- Lie's third theorem says that $(-)_{\infty}$ is essentially surjective

We will generalise this situation in two main ways:

- we will replace groups with categories
- we will use synthetic differential geometry to view the local and global objects in the same category:

$$
\mathit{Cat}_{\infty}(\mathcal{E}) \underset{(-)_{\infty}}{\overset{\hookrightarrow}{\leftarrows}} \mathit{Cat}(\mathcal{E}) \underset{\hookleftarrow}{\overset{(-)_{int}}{\rightleftarrows}} \mathit{Cat}_{int}(\mathcal{E}) \quad (\bot,\ \bot)
$$

Formal Group Laws

The Lie algebra of a Lie group constitutes a first order or linear approximation. In fact the jet part of a category will be more like an analytic approximation and hence analogous to a formal group law. The category of formal group laws is equivalent to *LieAlg*.

Definition 1.

A formal group law F of dimension n is an n-tuple $(F_1, ..., F_n)$ of power series in the indeterminates $X_1, ..., X_n, Y_1, ..., Y_n$ such that

$$F(0, \vec{Y}) = \vec{Y} \text{ , } F(\vec{X}, 0) = \vec{X} \text{ and } F(F(\vec{X}, \vec{Y}), \vec{Z}) = F(\vec{X}, F(\vec{Y}, \vec{Z}))$$

Example 2.

Given a Lie group (G, μ, e) choose a trivialisation $U \ni e$ and $g, h \in U$ such that $\mu(g, h) \in U$. If $g = \vec{X}$ and $h = \vec{Y}$ in the local coordinates then $\mu(\vec{X}, \vec{Y})$ is a formal group law in $\vec{X}$ and $\vec{Y}$.

Multi-object Lie Theory

The second advantage of using synthetic differential geometry is more subtle. First consider the following established multi-object generalisation of Lie theory.

Definition 3.
A *Lie groupoid* is a groupoid in *Man* such that the source and target maps are submersions.

Definition 4.
A *Lie algebroid* is a vector bundle $A \to M$ in *Man* together with a bundle homomorphism $\rho : A \to TM$ such that the space of sections $\Gamma(A)$ is a Lie algebra satisfying $(\forall X, Y \in \Gamma(A))(\forall f \in C^\infty(M))$:

$$[X, fY] = \rho(X)(f) \cdot Y + f[X, Y]$$

Multi-object Lie Theory

In the multi-object setting, we still have a full and faithful functor

$$LieGpd_{sc} \xrightarrow{T_e} LieAlgd$$

but it is not essentially surjective.

- For every Lie algebroid there is a topological groupoid that is the 'obvious' candidate for the integral of the algebroid (its Weinstein groupoid) but there can be obstructions to putting a smooth structure on it - see [Crainic and Fernandes 2003].

Idea: Enlarge the category of smooth spaces:-

- Differentiable Stacks [Tseng and Zhu 2006].
- Using Synthetic Differential Geometry.

Relevant Features of Synthetic Differential Geometry

In synthetic differential geometry, we work in a topos $\mathcal{E}$ such that:-

- There is a full and faithful embedding $Man \xrightarrow{\iota} \mathcal{E}$.
- There is a ring $R = \iota(\mathbb{R}) \in \mathcal{E}$.
- The object $D_k = \{x \in R : x^{k+1} = 0\}$ is not terminal, in fact the Kock-Lawvere axiom holds:

$$R^{k+1} \to R^{D_k}$$

$$(a_0, a_1, .., a_k) \mapsto (d \mapsto a_0 + a_1 d + ... + a_k d^k)$$

is an isomorphism.

- Using the Kock-Lawvere axiom, it is possible to show that

$$\iota(TG) = \iota G^D$$

and that formal group laws are precisely groups of the form (D^n_∞, μ) in $\mathcal{E}$.

- This means that the Lie algebra of a Lie group G can be described as

$$T_e(G) = \{\phi \in G^D : \phi(0) = e\}$$

with a bracket which is given by a certain infinitesimal commutator.

- Similarly since G is a smooth manifold there exist embeddings

$$\psi : D^n_\infty \rightarrowtail G \text{ such that } \psi(0) = e$$

Then to construct a formal group from ψ it only remains to check that the multiplication of G restricts to $im(\psi)$.

The Jet Part of a Category

Definition 5.
We write *SpecWeil* for the set of infinitesimal objects of the form:

$$\{(x_1, ..., x_n) : \bigwedge_{i=1}^{n}(x_i^{k_i} = 0) \wedge \bigwedge_{j=1}^{m}(p_j = 0)\}$$

for $n, m \in \mathbb{N}_{\geq 0}$, $k_i \in \mathbb{N}_{>0}$ and p_j are polynomials in the x_i.

Definition 6.
Let B be an object of $\mathcal{E}/M$. Let $a, b \in B$. Then we say that b is 'a jet away' from a iff

$$a \approx b \iff \bigvee_{D \in SpecWeil} \exists \phi \in B^D.\ \exists d \in D.\ (\phi(0) = a) \wedge (\phi(d) = b)$$

The Jet Factorisation System

Theorem 7.
Let $\mathbb{C} = (C, M, s, t, e, \mu)$ *be a category in* $\mathcal{E}$*. Then the subobject of* $s : C \to M$ *in* $\mathcal{E}/M$ *defined by*

$$C_\infty = \{c \in (s : C \to M) : esc \approx c\}$$

is closed under the composition μ *and hence defines a subcategory* $\iota_{\mathbb{C}}^\infty : \mathbb{C}_\infty \rightarrowtail \mathbb{C}$ *called the jet part of* $\mathbb{C}$*.*

Definition 8.
The category $Cat_\infty(\mathcal{E})$ is the full subcategory of $Cat(\mathcal{E})$ on the categories $\mathbb{C}$ for which $\iota_{\mathbb{C}}^\infty$ is an isomorphism.

Proposition 9.
The category $Cat_\infty(\mathcal{E})$ *is a coreflective subcategory of* $Cat(\mathcal{E})$ *such that for all* $\mathbb{C}, \mathbb{D} \in Cat(\mathcal{E})$ *the following arrow is an isomorphism*

$$(\iota_{\mathbb{C}}^\infty)^{\mathbb{D}_\infty} : \mathbb{C}_\infty^{\mathbb{D}_\infty} \to \mathbb{C}^{\mathbb{D}_\infty}$$

Paths in Categories

Definition 10.
The category $\mathbb{I}$ has underlying reflexive graph

$$\{(x,y) \in I^2 : x \leq y)\} \underset{\pi_1}{\overset{\pi_2}{\underset{\longleftarrow \Delta -}{\rightrightarrows}}} I$$

and the only possible composition. The category $\partial\mathbb{I}^2$ is the full subcategory of $\mathbb{I}^2$ on the boundary of I^2.

Definition 11.
Let $\mathbb{C}$ be a category in $\mathcal{E}$. Then

- A *path* in $\mathbb{C}$ is a functor $\mathbb{I} \to \mathbb{C}$
- A *jet path* in $\mathbb{C}$ is a functor $\mathbb{I}_\infty \to \mathbb{C}$

In fact there are the following endofunctors and natural transformations on $Cat(\mathcal{E})$

$$W \xLeftarrow{V} P \xRightarrow{L} 1_{Cat(\mathcal{E})}$$

where

- the category $P\mathbb{C}$ is the category of paths in $\mathbb{C}$ 'up to homotopy'
- the category $W\mathbb{C}$ is the category of jet paths in $\mathbb{C}$ 'up to homotopy'
- the natural transformation V is induced by the inclusion

$$\iota_{\mathbb{I}}^{\infty} : \mathbb{I}_{\infty} \rightarrowtail \mathbb{I}$$

- the natural transformation L is induced by the inclusion

$$(0,1) : \mathbf{2} \to \mathbb{I}$$

Connectedness of Categories

Definition 12.

Let $\mathbb{C}$ be a category in $\mathcal{E}$. Then $\mathbb{C}$ is path connected iff the following arrow is an epimorphism:

$$\mathbb{C}^{\mathbb{I}} \xrightarrow{\mathbb{C}^{(0,1)}} \mathbb{C}^{2}$$

and $\mathbb{C}$ is simply connected iff it is path connected and the following arrow is an epimorphism:

$$\mathbb{C}^{\mathbb{I}^2} \xrightarrow{\mathbb{C}^{\iota}} \mathbb{C}^{\partial\mathbb{I}^2}$$

Proposition 13.

If $\mathbb{C}$ is path connected then $L_{\mathbb{C}}$ is an epimorphism. If $\mathbb{C}$ is simply connected then $L_{\mathbb{C}}$ is an isomorphism.

Integral Complete Categories

In the synthetic setting we cannot assume that we always have solutions to time-dependent left-invariant vector fields.

Definition 14.

A category $\mathbb{C}$ in $\mathcal{E}$ is integral complete iff the following arrow is an isomorphism

$$\mathbb{C}^{\mathbb{I}} \xrightarrow{\mathbb{C}^{\iota^{\infty}_{\mathbb{C}}}} \mathbb{C}^{\mathbb{I}_\infty}$$

Proposition 15.

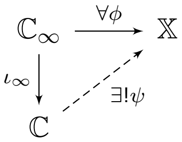
If $\mathbb{C}$ is integral complete then $V_{\mathbb{C}}$ is an isomorphism.

Proposition 16.

The category $Cat_{int}(\mathcal{E})$ of integral complete categories is a reflective subcategory of $Cat(\mathcal{E})$.

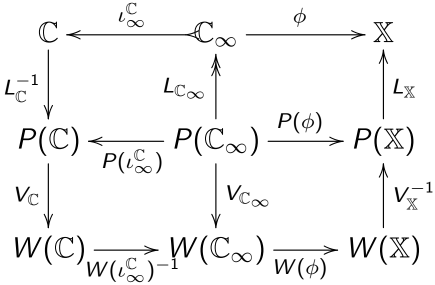
Synthetic Lie II

Theorem 17.
Let $\mathbb{C}$ be a simply-connected category such that the jet part $\mathbb{C}_\infty$ is path connected. Let $\mathbb{X}$ be an integral complete category. Then the following lifting property holds:



Proof: On the next slide.

Synthetic Lie II



The Jet Factorisation System

Definition 18.

A *jet-closed arrow in $\mathcal{E}$* is a monic arrow $m : A \rightarrowtail B$ such that the following square is a pullback:

$$\begin{array}{ccc} A^D & \xrightarrow{-\circ 0} & A \\ \downarrow{\scriptstyle m\circ -} & & \downarrow{\scriptstyle m\circ -} \\ B^D & \xrightarrow{-\circ 0} & B \end{array}$$

Definition 19.

A *jet-dense arrow in $\mathcal{E}$* is an arrow $f : X \to Y$ such that the following square is a pullback for every jet-closed m:

$$\begin{array}{ccc} A^Y & \xrightarrow{-\circ f} & A^X \\ \downarrow{\scriptstyle m\circ -} & & \downarrow{\scriptstyle m\circ -} \\ B^Y & \xrightarrow{-\circ f} & B^X \end{array}$$

The Jet Factorisation System

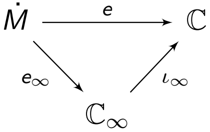
Definition/Proposition 20.

The jet factorisation system on $\mathcal{E}$ is given by

$$(L, R) = (\textit{jet-dense}, \textit{jet-closed})$$

and it induces a factorisation system on $Gpd(\mathcal{E})$.

Now for any category $\mathbb{C} = (C \rightrightarrows M) \in Cat(\mathcal{E})$ we can factorise the identity arrow:



If $Cat_\infty(\mathcal{E})$ is the subcategory of categories which have jet-dense identity map then the above factorisation induces a functor:

$$(-)_\infty : Cat(\mathcal{E}) \to Cat_\infty(\mathcal{E})$$

The Integral Factorisation System

An arrow $r : \mathbb{X} \to \mathbb{Y}$ is in the right class R_{int} (and is called integral closed) iff the following square is a pullback of categories:

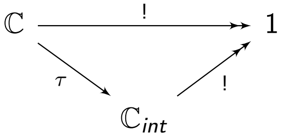
$$\begin{array}{ccc} \mathbb{X}^{\mathbb{I}} & \xrightarrow{\mathbb{X}^{\iota_\infty}} & \mathbb{X}^{\mathbb{I}_\infty} \\ \downarrow r^{\mathbb{I}} & & \downarrow r^{\mathbb{L}_\infty} \\ \mathbb{Y}^{\mathbb{I}} & \xrightarrow{\mathbb{Y}^{\iota_\infty}} & \mathbb{Y}^{\mathbb{I}_\infty} \end{array}$$

and an arrow $l : \mathbb{A} \to \mathbb{B}$ is in the left class L_{int} iff for all $r \in R_{int}$ the following square is a pullback:

$$\begin{array}{ccc} \mathbb{X}^{\mathbb{B}} & \xrightarrow{\mathbb{X}^{l}} & \mathbb{X}^{\mathbb{A}} \\ \downarrow r^{\mathbb{B}} & & \downarrow r^{\mathbb{A}} \\ \mathbb{Y}^{\mathbb{B}} & \xrightarrow{\mathbb{Y}^{l}} & \mathbb{Y}^{\mathbb{A}} \end{array}$$

Definition 21.

The integral completion $\mathbb{C}_{int}$ of a category $\mathbb{C}$ is the meditating object of the integral factorisation of the unique arrow $! : \mathbb{C} \to 1$:



An integral complete category is a category $\mathbb{C}$ for which $\tau : \mathbb{C} \to \mathbb{C}_{int}$ is an isomorphism and we write $Cat_{int}(\mathcal{E})$ for the full subcategory on integral complete categories.

Lemma 22.

The function $(-)_{int} : Cat(\mathcal{E}) \to Cat_{int}(\mathcal{E})$ extends to a functor.

Proof.

This is immediate by functoriality of factorisation. □

The Weinstein Groupoid

Definition 23.

The Weinstein groupoid $\bar{\mathbb{G}} = (\bar{G} \rightrightarrows M)$ of the groupoid $\mathbb{G} = (G \rightrightarrows M)$ has:

- object space M.
- arrow space the coequaliser:

$$\mathbb{G}^{\mathbb{O}_\infty} \underset{-\circ(\iota\circ\iota_1)}{\overset{-\circ(\iota\circ\iota_0)}{\rightrightarrows}} \mathbb{G}^{\mathbb{I}_\infty} \overset{q}{\twoheadrightarrow} \bar{\mathbb{G}}^2$$

- reflexive graph structure and multiplication induced by the factorisations of $ev_0, ev_1, const$ and μ. For example:

$$\begin{array}{ccccc}
\mathbb{G}^{\mathbb{O}_\infty +_* \mathbb{O}_\infty} & \underset{-\circ(\iota\circ\iota_1)}{\overset{-\circ(\iota\circ\iota_0)}{\rightrightarrows}} & \mathbb{G}^{\mathbb{I}_\infty +_* \mathbb{I}_\infty} & \twoheadrightarrow & \bar{\mathbb{G}}^2 \times_{G_0} \bar{\mathbb{G}}^2 \\
\downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \mu} \\
\mathbb{G}^{\mathbb{O}_\infty} & \underset{-\circ(\iota\circ\iota_1)}{\overset{-\circ(\iota\circ\iota_0)}{\rightrightarrows}} & \mathbb{G}^{\mathbb{I}_\infty} & \twoheadrightarrow & \bar{\mathbb{G}}^2
\end{array}$$

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