Synthetic Lie Theory

Lie's Second Theorem

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Main Idea

Idea

The passage from local data to global data in Lie theory is effected by the integration of 'infinitesimal' paths to macroscopic paths.

Example

- A path in a Lie algebra that lies in the domain of the exponential map induces a path in the Lie group starting at the identity.
- It turns out that paths in the Lie group starting at the identity are homotopic iff the corresponding paths in the Lie algebra are homotopic.
- If the Lie group is simply connected then homotopy classes of paths starting at the identity are the same thing as elements of the Lie group.

Summary

1. Multi-object Lie Theory
2. Path and Weinstein Categories
3. The Integral Factorisation System
4. Connectedness of Categories and Lie's Second Theorem

Multi-object Lie Theory

Definition

A *Lie groupoid* is a groupoid in *Man* such that the source and target maps are submersions.

Definition

A *Lie algebroid* is a vector bundle $A \to M$ in *Man* together with a bundle homomorphism $\rho : A \to TM$ such that the space of sections $\Gamma(A)$ is a Lie algebra satisfying $(\forall X, Y \in \Gamma(A))(\forall f \in C^\infty(M))$:

$$[X, fY] = \rho(X)(f) \cdot Y + f[X, Y]$$

The analogue of Lie's second theorem holds: the functor

$$LieAlgd \xleftarrow{T_e} LieGpd_{sc}$$

is full and faithful. However T_e is not essentially surjective.

Multi-object Lie Theory

For every Lie algebroid there is a topological groupoid called the Weinstein groupoid that is the ‘obvious’ candidate for the integral of the algebroid

$$\mathit{LieAlgd} \xrightarrow{W} \mathit{TopGpd} \dashrightarrow{??} \mathit{LieGpd}$$

but there can be obstructions to putting a smooth structure on it - see [Crainic and Fernandes 2003]. However if we enlarge our category of smooth spaces to a smooth topos $\mathcal{E}$, we can construct both a Weinstein category and a path category from every category:

$$\mathit{Cat}_{\infty}(\mathcal{E}) \underset{(-)_{\infty}}{\overset{\hookrightarrow}{\underset{\perp}{\longleftarrow}}} \mathit{Cat}(\mathcal{E}) \underset{W}{\overset{P}{\rightrightarrows}} \mathit{Cat}(\mathcal{E})$$

(Note that this is mildly more general than the classical case in which the domain of W is simply *LieAlgd*.)

Path and Weinstein Categories

Definition

The category $\mathbb{I}$ has underlying reflexive graph

$$\{(x,y) \in I^2 : x \leq y)\} \underset{\pi_1}{\overset{\pi_2}{\rightrightarrows}} I \quad (\xleftarrow{\Delta})$$

and the only possible composition. The category $\partial\mathbb{I}^2$ is the full subcategory of $\mathbb{I}^2$ on the boundary of I^2.

Definition

The Weinstein category $W\mathbb{C}$ of a category $\mathbb{C}$ has the same object space as $\mathbb{C}$ and arrow space given by the coequaliser

$$\mathbb{C}^{\mathbb{I}^2_\infty} \underset{\mathbb{C}^{\iota_2}}{\overset{\mathbb{C}^{\iota_1}}{\rightrightarrows}} \mathbb{C}^{\mathbb{I}_\infty {}_1{+}_0 \mathbb{I}_\infty} \longrightarrow\!\!\!\!\!\rightarrow (W\mathbb{C})^2$$

Path and Weinstein Categories

Definition

The path category $P\mathbb{C}$ of a category $\mathbb{C}$ has the same object space as $\mathbb{C}$ and arrow space given by the coequaliser

$$\mathbb{C}^{\mathbb{I}^2} \underset{\mathbb{C}^{\iota_2}}{\overset{\mathbb{C}^{\iota_1}}{\rightrightarrows}} \mathbb{C}^{\mathbb{I}_1 +_0 \mathbb{I}} \twoheadrightarrow (P\mathbb{C})^{\mathbf{2}}$$

In fact there are the following endofunctors and natural transformations on $Cat(\mathcal{E})$:

$$W \overset{V}{\Longleftarrow} P \overset{L}{\Longrightarrow} 1_{Cat(\mathcal{E})}$$

where the natural transformation V is induced by the inclusion

$$\iota_{\mathbb{I}}^{\infty} : \mathbb{I}_{\infty} \rightarrowtail \mathbb{I}$$

and the natural transformation L is induced by the inclusion

$$(0, 1) : \mathbf{2} \to \mathbb{I}$$

The Integral Factorisation System

In the synthetic setting we cannot assume that we always have solutions to time-dependent left-invariant vector fields.

Definition

An arrow $r : \mathbb{X} \to \mathbb{Y}$ is in the right class R_{int} (and is called integral closed) iff the following square is a pullback of categories:

$$\begin{array}{ccc} \mathbb{X}^{\mathbb{I}} & \xrightarrow{\mathbb{X}^{\iota_\infty}} & \mathbb{X}^{\mathbb{I}_\infty} \\ \downarrow r^{\mathbb{I}} & & \downarrow r^{\mathbb{L}_\infty} \\ \mathbb{Y}^{\mathbb{I}} & \xrightarrow{\mathbb{Y}^{\iota_\infty}} & \mathbb{Y}^{\mathbb{I}_\infty} \end{array}$$

and an arrow $l : \mathbb{A} \to \mathbb{B}$ is in the left class L_{int} iff for all $r \in R_{int}$ the following square is a pullback:

$$\begin{array}{ccc} \mathbb{X}^{\mathbb{B}} & \xrightarrow{\mathbb{X}^{l}} & \mathbb{X}^{\mathbb{A}} \\ \downarrow r^{\mathbb{B}} & & \downarrow r^{\mathbb{A}} \\ \mathbb{Y}^{\mathbb{B}} & \xrightarrow{\mathbb{Y}^{l}} & \mathbb{Y}^{\mathbb{A}} \end{array}$$

Integral Complete Categories

Definition

A category $\mathbb{C}$ in $\mathcal{E}$ is integral complete iff the following arrow is an isomorphism

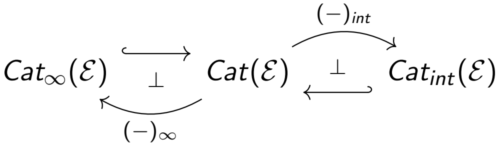
$$\mathbb{C}^{\mathbb{I}} \xrightarrow{\mathbb{C}^{\iota^{\infty}_{\mathbb{C}}}} \mathbb{C}^{\mathbb{I}_\infty}$$

Proposition

If $\mathbb{C}$ is integral complete then $V_{\mathbb{C}}$ is an isomorphism.

Proposition

The category $Cat_{int}(\mathcal{E})$ of integral complete categories is a reflective subcategory of $Cat(\mathcal{E})$.



Connectedness of Categories

Definition

Let $\mathbb{C}$ be a category in $\mathcal{E}$. Then $\mathbb{C}$ is path connected iff the following arrow is an epimorphism:

$$\mathbb{C}^{\mathbb{I}} \xrightarrow{\mathbb{C}^{(0,1)}} \mathbb{C}^{2}$$

and $\mathbb{C}$ is simply connected iff it is path connected and the following arrow is an epimorphism:

$$\mathbb{C}^{\mathbb{I}^2} \xrightarrow{\mathbb{C}^{\iota}} \mathbb{C}^{\partial\mathbb{I}^2}$$

Lemma

If $\mathbb{C}$ is path connected then $L_{\mathbb{C}}$ is an epimorphism. If $\mathbb{C}$ is simply connected then $L_{\mathbb{C}}$ is an isomorphism.

Lie's Second Theorem

Theorem

Let $\mathbb{C}$ be a simply-connected category such that the jet part $\mathbb{C}_\infty$ is path connected. Then $\iota_{\mathbb{C}}^{\infty} : \mathbb{C}_\infty \rightarrowtail \mathbb{C}$ is in the left class of the integral factorisation system.

Corollary

For all $\phi : \mathbb{C}_\infty \to \mathbb{X}$ there exists a unique lift ψ making

$$\begin{array}{ccc} \mathbb{C}_\infty & \xrightarrow{\forall \phi} & \mathbb{X} \\ {\scriptstyle \iota_\infty}\big\downarrow & \nearrow {\scriptstyle \exists! \psi} & \\ \mathbb{C} & & \end{array}$$

commute. Proof: On the next slide.

Lie's Second Theorem