A Version of Lie's Second Theorem for Categories

Matthew Burke

Category Theory Seminar, Cambridge University 2015-11-17

Main Idea

Explain how to formulate and prove Lie's second theorem for categories.
Describe how these constructions naturally generalise classical Lie theory.

Summary

1. Formal Group Laws and Classical Multi-object Lie Theory
2. Synthetic Differential Geometry and the Jet Part
3. The Path and Weinstein Categories
4. Connectedness and Lie's Second Theorem

Formal Group Laws

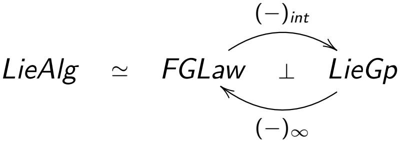
Definition

A formal group law F of dimension n is an n-tuple $(F_1, ..., F_n)$ of power series in the indeterminates $X_1, ..., X_n, Y_1, ..., Y_n$ such that

$$F(0, \vec{Y}) = \vec{Y} \ , \ F(\vec{X}, 0) = \vec{X} \text{ and } F(F(\vec{X}, \vec{Y}), \vec{Z}) = F(\vec{X}, F(\vec{Y}, \vec{Z}))$$

Example

Given a Lie group (G, μ, e) choose a trivialisation $U \ni e$ and $g, h \in U$ such that $\mu(g, h) \in U$. If $g = \vec{X}$ and $h = \vec{Y}$ in the local coordinates then $\mu(\vec{X}, \vec{Y})$ is a formal group law in $\vec{X}$ and $\vec{Y}$.



The adjunction $(-)_{int} \dashv (-)_\infty$ is an equivalence.

Multi-object Lie Theory

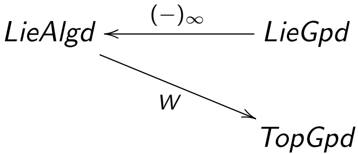
Definition

A *Lie groupoid* is a groupoid in *Man* such that the source and target maps are submersions.

Definition

A *Lie algebroid* is a vector bundle $A \to M$ in *Man* together with a bundle homomorphism $\rho : A \to TM$ such that the space of sections $\Gamma(A)$ is a Lie algebra satisfying $(\forall X, Y \in \Gamma(A))(\forall f \in C^\infty(M))$:

$$[X, fY] = \rho(X)(f) \cdot Y + f[X, Y]$$



The functor $(-)_\infty$ is full and faithful but not essentially surjective.

Relevant Features of Synthetic Differential Geometry

In synthetic differential geometry, we work in a topos $\mathcal{E}$ such that:-

- There is a full and faithful embedding $Man \xrightarrow{\iota} \mathcal{E}$.
- There is a ring $R = \iota(\mathbb{R}) \in \mathcal{E}$.
- The object $D_k = \{x \in R : x^{k+1} = 0\}$ is not terminal, in fact the Kock-Lawvere axiom holds:

$$R^{k+1} \to R^{D_k}$$

$$(a_0, a_1, .., a_k) \mapsto (d \mapsto a_0 + a_1 d + ... + a_k d^k)$$

is an isomorphism.

Definition

We write *SpecWeil* for the set of infinitesimal objects of the form:

$$\{(x_1, ..., x_n) : \bigwedge_{i=1}^{n} (x_i^{k_i} = 0) \wedge \bigwedge_{j=1}^{m} (p_j = 0)\}$$

for $n, m \in \mathbb{N}_{\geq 0}$, $k_i \in \mathbb{N}_{>0}$ and p_j are polynomials in the x_i.

The Jet Part of a Category

Definition

Let B be an object of $\mathcal{E}/M$. Let $a, b \in B$. Then we say that b is 'a jet away' from a iff

$$a \approx b \iff \bigvee_{D \in SpecWeil} \exists \phi \in B^D.\ \exists d \in D.\ (\phi(0) = a) \wedge (\phi(d) = b)$$

Theorem

Let $\mathbb{C} = (C, M, s, t, e, \mu)$ be a category in $\mathcal{E}$. Then the jet part $\mathbb{C}_\infty$ of $\mathbb{C}$ has arrow space defined by

$$C_\infty = \{c \in (s : C \to M) : esc \approx c\}$$

Proposition

The category $Cat_\infty(\mathcal{E})$ of all categories in $\mathcal{E}$ is coreflective in $Cat(\mathcal{E})$ and for all $\mathbb{C}, \mathbb{D} \in Cat(\mathcal{E})$ the arrow $(\iota^\infty_{\mathbb{C}})^{\mathbb{D}_\infty} : \mathbb{C}_\infty^{\mathbb{D}_\infty} \to \mathbb{C}^{\mathbb{D}_\infty}$ is an isomorphism.

The Path and Weinstein Categories

Definition

The category $\mathbb{I}$ has underlying reflexive graph

$$\{(x,y) \in I^2 : x \leq y)\} \underset{\pi_1}{\overset{\pi_2}{\underset{\longleftarrow \Delta -}{\longrightarrow}\longrightarrow}} I$$

and the only possible composition. The category $\partial\mathbb{I}^2$ is the full subcategory of $\mathbb{I}^2$ on the boundary of I^2.

Definition

Let $\mathbb{C}$ be a category in $\mathcal{E}$. Then

- A *path* in $\mathbb{C}$ is a functor $\mathbb{I} \to \mathbb{C}$
- A *jet path* in $\mathbb{C}$ is a functor $\mathbb{I}_\infty \to \mathbb{C}$

In fact there are the following endofunctors and natural transformations on $Cat(\mathcal{E})$

$$W \overset{V}{\Longleftarrow} P \overset{L}{\Longrightarrow} 1_{Cat(\mathcal{E})}$$

where

- the category $P\mathbb{C}$ is the category of paths in $\mathbb{C}$ 'up to homotopy'
- the category $W\mathbb{C}$ is the category of jet paths in $\mathbb{C}$ 'up to homotopy'
- the natural transformation V is induced by the inclusion

$$\iota_{\mathbb{I}}^{\infty} : \mathbb{I}_{\infty} \rightarrowtail \mathbb{I}$$

- the natural transformation L is induced by the inclusion

$$(0, 1) : \mathbf{2} \rightarrow \mathbb{I}$$

Integral Complete Categories

Definition

A category $\mathbb{C}$ in $\mathcal{E}$ is integral complete iff the following arrow is an isomorphism

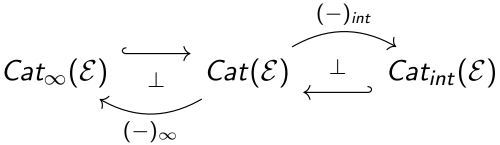
$$\mathbb{C}^{\mathbb{I}} \xrightarrow{\mathbb{C}^{\iota^{\infty}_{\mathbb{C}}}} \mathbb{C}^{\mathbb{I}_\infty}$$

Proposition

If $\mathbb{C}$ is integral complete then $V_{\mathbb{C}}$ is an isomorphism.

Proposition

The category $Cat_{int}(\mathcal{E})$ of integral complete categories is a reflective subcategory of $Cat(\mathcal{E})$.



Connectedness of Categories

Definition

Let $\mathbb{C}$ be a category in $\mathcal{E}$. Then $\mathbb{C}$ is path connected iff the following arrow is an epimorphism:

$$\mathbb{C}^{\mathbb{I}} \xrightarrow{\mathbb{C}^{(0,1)}} \mathbb{C}^{\mathbb{2}}$$

and $\mathbb{C}$ is simply connected iff it is path connected and the following arrow is an epimorphism:

$$\mathbb{C}^{\mathbb{I}^2} \xrightarrow{\mathbb{C}^{\iota}} \mathbb{C}^{\partial\mathbb{I}^2}$$

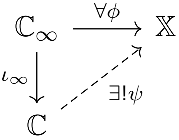
Lemma

If $\mathbb{C}$ is path connected then $L_{\mathbb{C}}$ is an epimorphism. If $\mathbb{C}$ is simply connected then $L_{\mathbb{C}}$ is an isomorphism.

Lie's Second Theorem

Theorem

For all $\phi : \mathbb{C}_\infty \to \mathbb{X}$ *there exists a unique lift* ψ *making*



commute.