Lie's Third Theorem in Synthetic Differential Geometry

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Summary

1. The Dubuc Topos and Germ Representability
2. Germs of Local Lie Groups and Lie's Third Theorem
3. Statment and Other Attempts
4. Closure Under Decomposition and Infinitesimal Closure

Lie's Third Theorem

$$\mathit{LieAlg} \xrightarrow{\simeq} \mathit{FGLaw} \xrightarrow{\simeq} \mathit{LGGerm} \underset{(-)_{\infty}}{\overset{(-)_{int}}{\underset{\perp}{\rightleftarrows}}} \mathit{LieGrp} \tag{1}$$

$$\mathit{LieAlgd} \longrightarrow ???? \longrightarrow \mathit{LGdGerm} \underset{(-)_{\infty}}{\overset{(-)_{int}}{\underset{\perp}{\rightleftarrows}}} \mathit{LieGrpd} \tag{2}$$

$$\mathit{SmthAlgd} \longrightarrow \mathit{Cat}_{jet}(\mathcal{E}) \longrightarrow \mathit{Cat}_{\infty}(\mathcal{E}) \underset{(-)_{\infty}}{\overset{(-)_{int}}{\underset{\perp}{\rightleftarrows}}} \mathit{Cat}_{int}(\mathcal{E}) \tag{3}$$

Theorem (Lie's Third Theorem)

If $\mathbb{K}$ is one of the local approximations then $(\mathbb{K}_{int})_{\infty} \cong \mathbb{K}$.

Germs of Local Lie Groups

Definition

A *local Lie group* consists of open sets $U_0, U \subset \mathbb{R}^n$ containing $\vec{0} \in \mathbb{R}^n$, a smooth map $\mu : U \times U \to \mathbb{R}^n$ and a smooth map $i : U_0 \to \mathbb{R}^n$ such that if $X, Y, Z \in U$ then:

- $\mu(X, 0) = X = \mu(0, X)$
- $X \in U_0 \implies \mu(X, i(X)) = \mu(i(X), X) = 0$
- $\mu(X, Y), \mu(Y, Z) \in U \implies \mu(X, \mu(Y, Z)) = \mu(\mu(X, Y), Z)$

Definition

A *germ of a local Lie group* is an equivalence class of local Lie groups where $G \sim H$ iff there exists a open neighbourhoods V_0 of $\vec{0} \in \mathbb{R}^n$ such that $\mu_G|V = \mu_H|V$ and $i_G|V_0 = i_H|V_0$.

Germ Representability in the Dubuc Topos

The *Dubuc topos* $\mathcal{E}$ is a topos such that:-

- There is a full and faithful embedding $Man \xrightarrow{\iota} \mathcal{E}$.
- There is a ring $R = \iota(\mathbb{R}) \in \mathcal{E}$.
- The object $D_k = \{x \in R : x^{k+1} = 0\}$ is not terminal, in fact the Kock-Lawvere axiom holds: $\alpha : R^{k+1} \to R^{D_k}$ defined by $(a_0, a_1, .., a_k) \mapsto (d \mapsto a_0 + a_1 d + ... + a_k d^k)$ is an isomorphism.

Definition

A *germ of a smooth function at* $\vec{x} \in \mathbb{R}^n$ is an equivalence class of smooth functions $\mathbb{R}^n \to \mathbb{R}$ where $f \sim g$ iff there exists a neighbourhood V of $\vec{x}$ such that $f|V = g|V$.

Definition

m_0 is the ideal of smooth maps $f : \mathbb{R}^n \to \mathbb{R}$ with null germ at $\vec{0}$.

Proposition

The object $[1, m] \cong \{x \in [1, (0)] : \neg\neg(x = 0)\}$.

The Nilradical and Jacobson Radical (Skipped)

In intuitionistic logic the following statements are not equivalent.

$$
\begin{array}{ccccc}
U(x) \vee \neg(x=0) & \longrightarrow & \neg(x=0) & \Longrightarrow & U(x) \\
\downarrow & & \downarrow & & \\
\neg U(x) \;\Longrightarrow\; (x=0) & \longrightarrow & \neg\neg(U(x) \vee (x=0)) & &
\end{array}
$$

hence we have four different kinds of field.

Definition

The *Jacobson radical* $J(R) = \{x \in R : \forall u \in R.\ 1 - ux \textit{ is inv.}\}$.
The *nilradical N(R)* is the set of all the nilpotent elements of R.
$N(R) \subset J(R)$ because $(1 - ud)(1 + ud + (ud)^2 + ... + (ud)^{k-1})$.

In a 'field of fractions' (top right definition above):

$$
\begin{aligned}
1 - ux \text{ is inv.} &\iff \neg(1 - ux = 0) \\
&\iff \neg(1 = ux) \\
&\iff \neg(x \text{ is inv.}) \\
&\iff \neg\neg(x = 0)
\end{aligned}
$$

Structure Required for Lie III

Definition

If $\mathbb{C} = C \rightrightarrows M$ is a category in $\mathcal{E}$ then the *infinitesimal part* $\mathbb{C}_\infty$ *of* $\mathbb{C}$ has object space M and arrow space $\{c \in C : \neg\neg(esc = c)\}$.

Definition

The *fundamental category* $\mathbb{I}$ on the unit interval $\mathbf{I}$ is

$$\mathbb{I}^{\mathbf{2}} := \{(a,b) \in \mathbf{I}^2 : a \leq b\} \underset{\pi_1}{\overset{\pi_0}{\overset{\rightarrow}{\underset{\rightarrow}{\xleftarrow{\Delta}}}}} \mathbf{I}$$

and the only possible composition.

Definition

If $\mathbb{C}$ is a category in $\mathcal{E}$ then *the integral completion* $\mathbb{C}_{int}$ *of* $\mathbb{C}$ is the pushout

$$\begin{array}{ccc} hom(\mathbb{I}_\infty, \mathbb{C}) \times \mathbb{I}_\infty & \xrightarrow{ev} & \mathbb{C} \\ \downarrow{\scriptstyle 1\times\iota} & & \downarrow{\scriptstyle \tau} \\ hom(\mathbb{I}_\infty, \mathbb{C}) \times \mathbb{I} & \xrightarrow{\alpha} & \mathbb{C}_{int} \end{array}$$

Attempts at Lie's Third Theorem

- For Lie algebroids the functor $(-)_{int} : \mathit{LieAlgs} \to \mathit{TopGrpd}$ doesn't factor through *LieGpd*. (Mention Tseng and Zhu.)
- with groupoids in synthetic differential geometry: counter example involving inverses
- with categories and nilpotents in synthetic differential geometry: still doesn't rule out all inverses! Similar counter example.

Closure under decomposition

Definition

$\beta : A \to B$ is *infinitesimally closed* iff β is a monomorphism and $\forall b \in B.\ [\neg\neg\,(\exists a \in A.\ \beta(a) = b) \implies \exists a_0 \in A.\ \beta(a_0) = b]$.

Lemma

Infinitesimally closed arrows are closed under pushout in $\mathcal{E}$.

Definition

A functor $\beta : \mathbb{A} \to \mathbb{B}$ is *closed under decomposition* iff $b \circ b' \in \beta(A) \implies b, b' \in \beta(A)$.

Theorem

In the following pushout τ is closed under decomposition.

$$\begin{array}{ccc} \mathbb{K}^{\mathbb{I}_\infty} \times \mathbb{I}_\infty & \xrightarrow{ev} & \mathbb{K} \\ \Big\downarrow{\scriptstyle \mathbb{K}^{\mathbb{I}_\infty} \times \iota_{\mathbb{I}}^{\infty}} & & \Big\downarrow{\scriptstyle \tau_K} \\ \mathbb{K}^{\mathbb{I}_\infty} \times \mathbb{I} & \xrightarrow{\alpha_K} & \mathbb{P} \end{array} \tag{4}$$

Main Theorem

Theorem

If $\mathbb{K}$ is an infinitesimal category and

$$\begin{array}{ccc} hom(\mathbb{I}_\infty, \mathbb{K}) \times \mathbb{I}_\infty & \xrightarrow{ev} & \mathbb{K} \\ \downarrow{\scriptstyle 1\times\iota} & & \downarrow{\scriptstyle \tau} \\ hom(\mathbb{I}_\infty, \mathbb{K}) \times \mathbb{I} & \xrightarrow{\alpha} & \mathbb{P} \end{array}$$

is a pushout in $Cat(\mathcal{E})$ then the arrow $\tau_\infty : \mathbb{K}_\infty \to \mathbb{P}_\infty$ is an isomorphism.

Proof.

Let $p \in P$ such that $\neg\neg(esp = p)$ and $p = L_0 \circ \ldots \circ L_n$ for $L_i \in Q$. First $\neg\neg(\forall i \in \{1, ..., n\}.\ L_i \in v(K))$ because $esp \in v(K)$ and τ is closed under decomposition. Therefore $\forall i \in \{1, .., n\}.\ \neg\neg(L_i \in v(K))$. Finally $\forall i \in \{1, ..., n\}.\ L_i \in v(K)$ because v is infinitesimally closed and therefore $p \in \tau(K)$ as required. $\square$

Germs of Local Lie Groups in SDG

If time at end: formulate germs of local Lie groups in SDG.