K-rings and the Serre-Swan Theorem via Projection Vector Bundles

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Abstract

A slightly non-standard treatment of the elementary theory of vector bundles in terms of projection operators. We see how the K-ring is defined in this context and how the Serre-Swan theorem arises in a formal manner from an isomorphism between simpler categories.

Contents

1 Projection Vector Bundles and the K-semiring

Consider the category $\mathcal{L}$ of finite-dimensional normed real vector spaces with linear maps as arrows. We first observe that we can put a topological structure on spaces of arrows between any two objects $V, W \in \mathcal{L}$:

Definition 1.1. *The* operator norm *on the spaces $\mathcal{L}(V, W)$ is defined by:*

$$||\alpha||_{op} := sup\{||\alpha v|| : (v \in V) \wedge (||v|| = 1)\}$$

This induces a topology on the subspace $\mathcal{P}(V)$ of idempotent arrows (projection operators) on an object $V \in \mathcal{L}$:

$$\mathcal{P}(V) := \{\pi \in \mathcal{L}(V, V) : \pi^2 = \pi\}$$

The standard definition of a vector bundle over X starts with a continuous family of vector spaces over X and insists that this family is locally trivial. The following definition coincides (up to the isomorphisms in remark 4 below) with the standard one when X is a compact Hausdorff space.

Definition 1.2. *A* (projection) vector bundle over a topological space X *is a continuous map*

$$\pi : X \to \mathcal{P}(V)$$

Definition 1.3. *A* momorphism between vector bundles $\pi : X \to \mathcal{P}(V)$ and $\eta : X \to \mathcal{P}(W)$ *is a continuous map*

$$\Phi : X \to \mathcal{L}(V, W)$$

such that for all $x \in X$ *we have that* $\eta_x \circ \Phi_x \circ \pi_x = \Phi_x$*. Composition is defined using composition in* $\mathcal{L}$*:*

$$(\Phi \circ \Psi)_x = \Phi_x \circ \Psi_x$$

The identity map on the object π *is the* arrow *defined by* π*. Indeed vector bundles over* X *form a category* $VB(X)$*.*

Remark.

1. This is the idempotent completion of a category we will identify in section 3.
2. All projection vector bundles are locally trivial, but the converse only holds when X is a compact Hausdorff space.
3. We inherit direct sums, tensor products, images, coimages etc. from $\mathcal{L}$. There is a little to check here (namely that the result gives a continuous map).
4. We do not rule out oblique projections. However if π_x, ν_x are two objects of $VB(X)$ with the same image (i.e. that $\pi_x \nu_x = \nu_x$ and $\nu_x \pi_x = \pi_x$), then there is the isomorphism:

$$\pi_x \underset{\nu_x}{\overset{\pi_x}{\rightleftarrows}} \nu_x$$

Example 1.1. The tangent bundle over S^2:

$$\begin{aligned} S^2 &\to \mathcal{P}(\mathbb{R}^3) \\ \mathbf{x} &\mapsto proj_{\mathbf{x}^\perp} = (\mathbf{y} \mapsto \mathbf{y} - (\mathbf{y} \cdot \mathbf{x})\mathbf{x}) \end{aligned}$$

Example 1.2. The trivial 2-dimensional bundle over S^2:

$$\begin{aligned} S^2 &\xrightarrow{id_2} \mathcal{P}(\mathbb{R}^2) \\ \mathbf{x} &\mapsto 1_{\mathbb{R}^2} \end{aligned}$$

Remark. But why didn't we choose to use a larger ambient space? That is to say, why not define the trivial bundle as:

$$\begin{aligned} S^2 &\xrightarrow{id_2'} \mathcal{P}(\mathbb{R}^3) \\ \mathbf{x} &\mapsto proj_{\mathbf{e}^\perp} = (\mathbf{y} \mapsto \mathbf{y} - (\mathbf{y} \cdot \mathbf{e})\mathbf{e}) \end{aligned}$$

where $e = (1,0,0)$? However, the two options turn out to be isomorphic:

$$
\begin{aligned}
id_2' &\to id_2 \\
X &\to \mathcal{L}(\mathbb{R}^3, \mathbb{R}^2) \\
\mathbf{x} &\mapsto proj_{\mathbf{e}_\perp}
\end{aligned}
$$

which has as inverse the inclusion of $\mathbf{e}_\perp$ into $\mathbb{R}^3$.

The next two examples tell us the same information about the topological space X but are not isomorphic as vector bundles.

Example 1.3. The Möbius bundle:

$$
\begin{aligned}
S^1 &\to \mathcal{P}(\mathbb{R}^2) \\
\theta &\mapsto proj_{\langle \frac{\theta}{2} \rangle}
\end{aligned}
$$

Visually, this is a line in $\mathbb{R}^2$ rotating about the origin at 'half speed'.

Example 1.4. The 2-dimensional Möbius bundle:

$$
\begin{aligned}
S^1 &\to \mathcal{P}(\mathbb{R}^3) \\
\theta &\mapsto proj_{\langle \frac{\theta}{2}, (0,0,1) \rangle}
\end{aligned}
$$

Visually, this is a plane in $\mathbb{R}^3$ rotating about a line through the origin at 'half speed'.

The second example doesn't tell us any more than the first about the topological structure of $X = S^1$ but simply 'carries around' one extra dimension. Surely we would want to identify two bundles which differ only by such extraneous dimensions? Thus we arrive at the definition of reduced K-semiring:

Definition 1.4. *The* reduced K-semiring $K(X)$ of a topological space X *is the set of (topoogical) equivalence classes of vector bundles over X with direct sum as multiplication and tensor product as multiplication. Two vector bundles π, η are* topologically equivalent *precisely when:*

$$\pi \approx \eta \iff (\exists m, n)\ \pi \oplus id_n = \eta \oplus id_m$$

Remark. In particular, if we have a vector bundle $\pi : X \to \mathcal{P}(V)$ and a 1-dimensional subspace $W \subseteq V$ such that π_x fixes W for all x, then:

$$\pi \approx \pi|_{W^\perp}$$

so examples 1.3 and 1.4 are conflated.

2 Quotients and the K-ring

Recall that all objects in $VB(X)$ have an underlying linear endomorphism. The following lemma tells us when the pointwise subtraction is again an object of $VB(X)$.

Lemma 2.1. *If π_1, π_2 are objects in $VB(X)$ such that:*

$$\pi_2 \leq \pi_1 \iff \pi_2\pi_1 = \pi_2$$

holds then the difference $\pi_1 - \pi_2$ is again an object of $VB(X)$.

Proof.

$$\begin{aligned}(\pi_1 - \pi_2)(\pi_1 - \pi_2) &= \pi_1 - \pi_1\pi_2 - \pi_2\pi_1 + \pi_2 \\ &= \pi_1 - \pi_2 - \pi_2 + \pi_2 \\ &= (\pi_1 - \pi_2)\end{aligned}$$

□

Definition 2.1. *The* quotient vector bundle of π_1 by π_2 (for $\pi_2 \leq \pi_1 \iff \pi_2\pi_1 = \pi_2$) *is the projection $(\pi_1 - \pi_2)$. The* complement $\pi^\perp$ of π *is the quotient $\pi^\perp := (id - \pi)$.*

Definition 2.2. *The K-ring $K(X)$ of a topological space X is the K-semiring of X with the additive inverse of an object π given by the complement:*

$$\pi \oplus \pi^\perp \cong id_n \approx id_0$$

3 Idempotent Completions and the Serre-Swan Theorem

Consider the category $C(X)$-Mod where $C(X)$ is the ring of continuous functions from X to $\mathbb{R}$. We begin by recording the isomorphism which is the core of the Serre-Swan theorem.

Lemma 3.1. *There is an isomorphism of categories between the full subcategory $\mathcal{C}_{\mathbb{T}}$ of $C(X)$-Mod whose objects are free $C(X)$-modules and the full subcategory $TB(X)$ of $VB(X)$ whose objects are the trivial vector bundles.*

Proof. Every trivial vector bundle is of the form:

$$\begin{aligned}X &\xrightarrow{id_n} \mathcal{P}(\mathbb{R}^n) \\ \mathbf{x} &\mapsto 1_{\mathbb{R}^n}\end{aligned}$$

So we have a bijection on objects:

$$\begin{aligned}TB(X) &\xrightarrow{\phi} \mathcal{C}_{\mathbb{T}} \\ id_n &\mapsto C(X)^n\end{aligned}$$

The set of maps from id_n to id_m is $Top(X, \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m))$ which is in bijection with $C(X)$-module homomorphisms between $C(X)^n$ and $C(X)^m$:

$$\begin{aligned} Top(X, \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)) &\cong Top(X, \mathbb{R}^{n \cdot m}) \\ &\cong Top(X, \mathbb{R})^{n \cdot m} \\ &\cong C(X)^{n \cdot m} \\ &\cong C(X)\text{-Mod}(C(X)^n, C(X)^m) \end{aligned}$$

This isomorphism takes

$$(\Phi(x))_{ij} \mapsto (\Phi_{ij})(x)$$

and so is clearly functorial. □

Now the idempotent completion can be characterised as the closure under retracts of the image of the Yoneda embedding in the presheaf category. The following two statements should be clear:

- The idempotent completion of $\mathcal{C}_{\mathbb{T}}$ is the full subcategory $C(X)\text{-Mod}^{f.g.}_{proj}$ of $C(X)$-Mod whose objects are the finitely generated projective modules.
- The idempotent completion of $TB(X)$ is the category $VB(X)$. (Note that because every object has a complement it is the retract of a trivial bundle.)

So we have shown that:

Theorem 3.1. *(Serre-Swan) There is an equivalence of categories:*

$$C(X)\text{-}Mod^{f.g.}_{proj} \simeq VB(X)$$

References

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